Fundamentals of structural engineering pdf

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Dr. Faraji has close to three decades of teaching, research, publication, and consulting experience and has taught a wide range of courses at the undergraduate and at the graduate level, such as Concrete Design, Steel Design, Bridge Design, Seismic Design, Concrete Design, Finite Elements, Structural Dynamics, and Behavior of Structures. Her professional work involves the analysis, design (using the latest design codes), and computer modeling of a wide range of structures. She has worked on more than 30 bridges and on aircraft, elevators, pontoons, ramps, concrete parking garages, domes, culverts, retaining walls, and steel and concrete frame buildings. Jerome J. Connor · Susan Faraji Fundamentals of Structural Engineering Second Edition Fundamentals of Structural Engineering Jerome J. Connor • Susan Faraji Fundamentals of Structural Engineering Second Edition Jerome J. Connor Department of Civil & Environmental Engineering Massachusetts Institute of Technology Cambridge, MA, USA Susan Faraji Department of Civil & Environmental Engineering University of Massachusetts-Lowell Lowell, MA, USA ISBN 978-3-319-24329-0 ISBN 978-3-319-24331-3 (eBook) DOI 10.1007/978-3-319-24331-3 Library of Congress Control Number: 2015958840 Springer Cham Heidelberg New York Dordrecht London Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com) Preface The first edition considered only linear elastic behavior of structures. This assumption is reasonable for assessing the structural response in the early stage of design where one is attempting to estimate design details. As a design progresses, other critical behavioral issues need to be addressed. The first issue concerns geometric nonlinearity which results when a flexible member is subjected to axial compression loading as well as transverse loading. This combination causes a loss in axial stiffness for the member, which may result in a loss in stability for the structural system. Euler buckling is an example of this type of nonlinear behavior. The second issue is related to the behavior of the material used to fabricate structural members. Steel and concrete are the most popular materials for structural applications. These materials have a finite elastic range, i.e., they behave elastically up to a certain stress level. Beyond this level, their stiffness decreases dramatically and they experience significant deformation that remains when the specimen is unloaded. This deformation is referred to as "inelastic deformation." The result of this type of member behavior is the fact that the member has a finite load carrying capacity. From a structural system perspective, it follows that the structure has a finite load capacity. Given the experience with recent structural failures, structural engineers are now being required to estimate the "limit" capacity of their design using inelastic analysis procedures. Computer-based analysis is essential for this task. We have addressed both issues in this edition. Geometric nonlinearity is basically a displacement issue, so it is incorporated in Chap. 10. We derive the nonlinear equations for a member; develop the general solution, specialize the solutions for various boundary conditions; and finally present the generalized nonlinear "member" equations which are used in computer-based analysis methods. Examples illustrating the effect of coupling between compressive axial load and lateral displacement (P-delta effect) are included. This treatment provides sufficient exposure to geometric nonlinearity that we feel is necessary to prepare the student for professional practice. Inelastic analysis is included in Part III which deals with professional practice; we have added an additional chapter focused exclusively on inelastic analysis. We start by reviewing the basic properties of structural steel and concrete and then establish the expressions for the moment capacity of beams. We use these results together with some simple analytical methods to establish the limit loading for some simple beam and frames. For complex structures, one needs to resort to computer-based procedures. We describe a finite element-based method that allows one to treat the nonlinear load displacement behavior and to estimate the limiting load. This approach is referred to as a "pushover" analysis. Examples illustrating pushover analyses of frames subjected to combined gravity and seismic loadings are included. Just as for the geometric nonlinear case, our objective is to provide sufficient exposure to the material so that the student is "informed" about the nonlinear issues. One can gain a deeper background from more advanced specialized references. Aside from these two major additions, the overall organization of the second edition is similar to the first edition. Some material that we feel is obsolete has been deleted (e.g., conjugate beam), and other materials such as force envelopes have been expanded. In general, we have tried to place more emphasis on computer base approaches since professional practice is moving in that direction. However, we still place the primary emphasis on developing a fundamental understanding of structural behavior through analytical solutions and computer-based computations. Audience The intended audience of this book is that of students majoring in civil engineering or architecture who have been exposed to the basic concepts of engineering mechanics and mechanics of materials. The book is sufficiently comprehensive to be used for both undergraduate and higher level structures subjects. In addition, it can serve students as a valuable resource as they study for the engineering certification examination and as a reference later in their careers. Practicing professionals will also find the book useful for self-study, for review for the professional registration examination, and as a reference book. Motivation The availability of inexpensive digital computers and user-friendly structural engineering software has revolutionized the practice of structural engineering. Engineers now routinely employ computer-based procedures throughout the various phases of the analysis and design detailing processes. As a result, with these tools engineers can now deal with more complex structures than in the past. Given that these tools are now used in engineering practice, the critical question facing faculty involved in the teaching of structural engineering is "How the traditional teaching paradigm should be modified for the computer age?" We believe that more exposure to computer-based analysis is needed at an early stage in the course development. However, since the phrase "garbage in garbage out" is especially relevant for computer-based analysis, we also believe that the student needs to develop, through formal Preface Preface vi training in analysis methodology, the ability to estimate qualitatively the behavior of a structure subjected to a given loading and to confirm qualitative estimates with some simple manual computations. Based on a review of the current structural engineering academic literature, it appears that the current set of undergraduate textbooks are focused mainly on either (1) teaching manual analysis methods and applying them to simple idealized structures or (2) reformulating structural analysis methods in terms of matrix notation. The first approach is based on the premise that intuition about structural behavior is developed as one works through the manual computations, which, at times, may seem exhaustive. The second approach provides the basis for developing and understanding computer software codes but does not contribute toward developing intuition about structural behavior. Clearly there is a need for a text that provides a balanced treatment of both classical and modern computer-based analysis methods in a seamless way and also stresses the development of an intuitive understanding of structural behavior. Engineers reason about behavior using simple models and intuition that they have acquired through problem-solving experience. The approach adopted in this text is to develop this type of intuition through computer simulation which allows one to rapidly explore how the structure responds to changes in geometry and physical parameters. We believe this approach better prepares the reader for the practice of structural engineering. Objectives Structural engineers have two major responsibilities during the design process. First, they must synthesize the structural system, i.e., select the geometry and the type of structural members that make up the structure. Second, they must size the members such that the structure can comfortably support the design loading. Creating a structural concept requires a deep knowledge of structural behavior. Sizing the members requires information about the internal forces resulting from the loading. These data are acquired through intelligent application of analysis methods, mainly computer-based methods. With these responsibilities in mind, we have selected the following objectives for this book: • Develop the reader's ability to analyze structures using manual computational procedures. • Educate the reader about structural behavior. We believe that a strong analytical background based on classical analysis methodology combined with computer simulation facilitates the development of an understanding of structural behavior. • Provide the reader with an in-depth exposure to computer-based analysis methods. Show how computer-based methods can be used to determine, with minimal effort, how structures respond to loads and also how to establish the extreme values of design variables required for design detailing. viii • Develop the reader's ability to validate computer-based predictions of structural response. • Provide the reader with idealization strategies for reducing complex structures to simple structural models. • Develop an appreciation for and an awareness of the limitations of using simple structural models to predict structural behavior through examples which illustrate behavioral trends as structures become more complex. Organization We have organized this text into three parts. Parts I and II are intended to provide the student with the necessary computational tools and also to develop an understanding of structural behavior by covering analysis methodologies, ranging from traditional classical methods through computer-based methods, for skeletal-type structures, i.e., structures composed of one-dimensional slender members. Part I deals with statically determinate structures; statically indeterminate structures are covered in Part II. Certain classical methods which we consider redundant have been omitted. Some approximate methods which are useful for estimating the response using hand computations have been included. Part III is devoted to structural engineering issues for a range of structures frequently encountered in practice. Emphasis is placed on structural idealization, how one identifies critical loading patterns, and how one generates the extreme values of design variables corresponding to a combination of gravity, live, wind, earthquake loading, and support settlement using computer software systems. Brief descriptions of the subject content for each part are presented below. Part I discusses statically determinate structures. We start with an introduction to structural engineering. Statically determinate structures are introduced next. The treatment is limited to linear elastic behavior and static loading. Separate chapters are devoted to different skeletal structural types such as trusses, beams, frames, cables, curved members, footings, and retaining walls. Each chapter is self-contained in that all the related analysis issues for the particular structural type are discussed and illustrated. For example, the chapter on beams deals with constructing shear and moment diagrams, methods for computing the deflection due to bending, influence lines, force envelopes, and symmetry properties. We find it convenient from a pedagogical perspective to concentrate the related material in one location. It is also convenient for the reader since now there is a single source point for knowledge about each structural type rather than having the knowledge distributed throughout the text. We start with trusses since they involve the least amount of theory. The material on frames is based on beam theory, so it is logical to present it directly after beam theory. Cables and curved members are special structural types that generally receive a lower priority, due to time constraints, when selecting a syllabus. We have included these topics here, as Preface Preface ix well as a treatment of footings and retaining walls, because they are statically determinate structures. We revisit these structures later in Part III. Part II presents methods for analyzing statically indeterminate structures and applies these methods to a broad range of structural types. Two classical analysis methods are described, namely, the force (also referred to as the flexibility) method and the displacement (or stiffness) method. We also present some approximate analysis methods that are based on various types of force and stiffness assumptions. These methods are useful for estimating the structural response due to lateral loads using simple hand computations. Lastly, we reformulate the traditional displacement method as a finite element method using matrix notation. The finite element formulation (FEM) is the basis of most existing structural analysis software packages. Our objectives here are twofold: first, we want to enable the reader to be able to use FEM methods in an intelligent way, and second, we want the reader to develop an understanding of structural behavior by applying analysis methods to a broad range of determinate and indeterminate skeletal structures. We believe that using computer analysis software as a simulation tool to explore structural behavior is a very effective way of building up a knowledge base of behavioral modes, especially for the types of structures commonly employed in practice. Part III discusses typical structural engineering problems. Our objective here is to expose the reader to a select set of activities that are now routinely carried out by structural engineers using structural engineering software. These activities are related to the approach followed to establish the "values" for the design variables. Defining these values is the key step in the engineering design process; once they are known, one can proceed to the design detailing phase. Specific chapters deal with horizontal structures such as multi-span girder, arch, and cable-stayed bridge systems, modeling of three-dimensional vertical structures subjected to lateral loading, and vertical structures such as low- and high-rise buildings subjected to gravity loading. The topics cover constructing idealized structural models, establishing the critical design loading patterns for a combination of gravity and live loading, using analysis software to compute the corresponding design values for the idealized structures, defining the lateral loading due to wind and earthquake excitation for buildings, and estimating the three-dimensional response of low-rise buildings subjected to seismic and wind loadings. Course Suggestions The following suggestions apply for students majoring in either civil engineering or architecture. Depending on the time available, we suggest organizing the material into either a two-semester or a three-semester sequence of subjects. x Our recommendations for the three-semester sequence are as follows: Structures I The goal of this subject is to provide the skills for the analysis of statically determinate trusses, beams, frames, and cables and to introduce some computer-based analysis methods. Chapters 1, 2, part of 3, part of 4, and the first part of 5 Structures II The objective of this subject are to present both classical and computerbased analysis methods for statically indeterminate structures such as multispan beams, gable frames, arches, and cable-stayed structures subjected to various loadings. The emphasis is on using analysis methods to develop an understanding of the behavior of structures. Chapters 9, 10, 11, 12, 6, and the last part of 5 Structures III This subject is intended to serve as an introduction to the practice of structural engineering. The material is presented as case studies for the two most common types of structures, bridges, and buildings. Issues such as geometrical configurations, idealized structural models, types and distribution of loadings, critical loading patterns, and determination of the values of the design variables such as the peak moment in a beam, force envelopes, and inelastic behavior are discussed. Both the superstructure and the substructure components are considered. Extensive use of computer software is made throughout the subject. Recitation classes dealing with the design detailing of steel and concrete elements can be taught in parallel with the lectures. Chapters 13, 14, 15, 16, 7, and 8 The makeup of the two-semester sequence depends on how much background in mechanics and elementary structures the typical student has and the goal of the undergraduate program. One possibility is to teach Structures I and II described above. Another possible option is to combine Structures I and II into a single subject offering together with Structures III. A suggested combined subject is listed below. Structures (Combined I + II) Chapters 3, 4 (partial), 9 (partial), 10, 11, and 12 Preface Preface xi Features of the Text Organization by Structural Type The chapters are organized such that an individual chapter contains all the information pertaining to a particular structural type. We believe this organization facilitates access to information. Since the basic principles are generic, it also reinforces these principles throughout the development of successive chapters. Classical Analysis Methods In-depth coverage of classical analysis methods with numerous examples helps students learn fundamental concepts and develop a "feel" and context for structural behavior. Analysis by Hand Computation The book helps teach students to do simple hand computing, so that as they move into doing more complex computational analysis, they can quickly check that their computer-generated results make sense. Gradual Introduction of Computer Analysis The text provides students with a gradual transition from classical methods to computational methods, with examples and homework problems designed to bring students along by incorporating computational methods when most appropriate to in-depth coverage of finite element methods for skeletal structures. Example Problems Example problems in each chapter illustrate solutions to structural analysis problems, including some problems illustrating computer analysis. Most of the example problems are based on real scenarios that students will encounter in professional practice. Units Both SI and customary US units are used in the examples and homework problems. Homework Problems that Build Students' Skills An extensive set of homework problems for each chapter provides students with the practice they need to hone the concepts and skills developed in the chapters. The xii Preface difficulty level is varied so that students can build confidence by starting with simple problems and advancing toward more complex problems. Comprehensive Breadth and Depth, Practical Topics The comprehensive breadth and depth of this text means it may be used for two or more courses, so it is useful to students for their courses and as a professional reference. Special topics such as the simplifications associated with symmetry and antisymmetry, arch-type structures, and cable-stayed structures are topics that a practicing structural engineer needs to be familiar with. Cambridge, MA Lowell, MA Jerome J. Connor Susan Faraji Acknowledgments We would like to thank our spouses Barbara Connor and Richard Hennessey for their patience and moral support over the seemingly endless time required to complete this text. We are most appreciative. We would also like to thank our colleagues and students who provided us with many valuable suggestions concerning the content and organization of the text. We are especially indebted to Dr. Moneer Tewfik and Dr. Carlos Brebbia for their constructive criticisms and enthusiastic support as the text was evolving into its final form. xiii Contents Part I Photo Credits Chapter 1 Fig. 1.1c Offshore Platform, Brazil. This image was produced by Ageˆncia Brasil, a public Brazilian news agency and published under a Creative Commons Attribution License. It was accessed in February 2012 from Oil_platform_P-51_(Brazil).jpg Fig. 1.1a Skyscraper under construction in Kutuzovsky Prospekt, Moscow, Russia. This image, created by Denghu, is licensed under a Creative Commons Attribution 3.0 Unported License. The image was accessed in April 2012 from Skyscraper_ Kutuzovsky_Prospekt_Moscow.jpg Fig. 1.8 Millau Viaduct, Author: Delphine DE ANDRIA Date: 18.11.2007, from FreeMages. Accessed May 2012 from browse/photo-916-millau-viaduct.html. This work is licensed under a Creative Commons Attribution 3.0 Unported License. Chapter 2 Fig. 2.13 Three-dimensional truss of roof system. This image of the lead shot tower in Melbourne, Australia, created by freeaussiestock.com, is licensed under a Creative Commons Attribution 3.0 Unported License. The image was accessed in February 2012 from ria/Melbourne/slides/shot_tower.htm Chapter 5 Fig. 5.1 Clifton Suspension bridge, England. Picture taken by Adrian Pingstone in October 2003 and placed in the public domain. Accessed in February 2012 from Clifton.bridge. arp.750pix.jpg xxv xxvi Fig. 5.5 Munich Olympic stadium, view from Olympic Tower. Picture taken by Arad Mojtahedi in July 2008 and placed in the public domain. Accessed in May 2012 from Olympiastadion_ Muenchen.jpg Fig. 5.21 Millau Viaduct, Author: Delphine DE ANDRIA, Date: 18.11.2007, from FreeMages. Accessed in May 2012 from browse/photo-916-millau-viaduct.html. This work is licensed under a Creative Commons Attribution 3.0 Unported License. Chapter 6 Fig. 6.5 Alcantara Toledo Bridge, Puente de Alca´ntara, Toledo, Spain. This image was originally posted to Flickr on December 19, 2006 and published under a Creative Commons Attribution License. It was accessed in February 2012 from Puente_ Alcantara_toledo.jpg Fig. 6.7 Eads Bridge, USA. This image was originally posted to Flickr by Kopper at http://flickr.com/photos//2745897992. It was reviewed on January 18, 2011 (2011-01-18) by the FlickreviewR robot and confirmed to be licensed under the terms of the cc-by-2.0 (Creative Commons Attribution 2.0). It was accessed in February 2012 from http:// commons.wikimedia.org/wiki/File:Eads_Bridge-1.jpg Fig. 6.8 Salginatobel Bridge, Switzerland. Bild (Figure) 1.4-45 Ansicht der Salginatobelbru¨cke. P. 81 in Mehlhorn, G. et al. "Bru¨ckenbau auf dem Weg vom Altertum zum modernen Bru¨ckenbau," in Handbuch Bru¨cken. Gerhard Mehlhorn (Ed). Berlin, Heidelberg. Springer-Verlag (2010) Fig. 6.9a New Gorge Arch, West Virginia. This image was originally posted to Flickr by nukeit1 at http://flickr.com/photos// 244750516. It was reviewed on November 14, 2007 (2007-11-14) by the FlickreviewR robot and confirmed to be licensed under the terms of the cc-by-2.0 (Creative Commons Attribution 2.0). It was accessed in February 2012 from New_River_Gorge_ Bridge_West_Virginia_244750516.jpg Chapter 8 Fig. 8.4 Gravity retaining wall. Courtesy of HNTB Corporation, 31 St. James Avenue, Suite 300 Boston, MA 02116, USA Chapter 13 Fig. 13.1a Multi-span curved steel box girder bridge. Courtesy of HNTB Corporation, 31 St. James Avenue, Suite 300, Boston, MA 02116, USA Photo Credits Photo Credits xxvii Fig. 13.1c The John James Audubon Bridge crossing the Mississippi River. This image is credited to the Louisiana TIMED Managers and was accessed in April 2012 from Audubon_ Bridge2.jpg Fig. 13.34 Typical cable-stayed scheme. This work has been released into the public domain by its author, Kelly C. Cook. This applies worldwide. The image was accessed in February 2012 from wiki/File:Sunshine_Skyway_from_Tampa_Bay.jpeg Problem 13.9 Puente del Alamillo in Seville, Spain. This work has been released into the public domain by its author, Consorcio Turismo Sevilla. This applies worldwide. The image was accessed in March 2012 from Puente_del_Alamillo.jpg Part I Statically Determinate Structures A structure is an assemblage of components which are connected in such a way that the structure can withstand the action of loads that are applied to it. These loads may be due to gravity, wind, ground shaking, impact, temperature, or other environmental sources. Structures are everywhere in the built environment. Buildings, bridges, tunnels, storage tanks, and transmission lines are examples of a "structure." Structural engineering is the discipline which is concerned with identifying the loads that a structure may experience over its expected life, determining a suitable arrangement of structural members, selecting the material and dimensions of the members, defining the assembly process, and lastly monitoring the structure as it is being assembled and possibly also over its life. In Part I, we first present an overview of structural engineering so that the reader can develop an appreciation for the broad range of tasks that structural engineers carry out and the challenges that they face in creating structures which perform satisfactorily under the loadings that they are subjected to. We then discuss a particular subgroup of structures called statically determinate structures. This subgroup is relatively easy to deal with analytically since only equilibrium concepts are involved. Also, most structures belong to this category. Trusses, beams, frames, cables, curved members, shallow foundations, and vertical retaining walls are described in separate chapters. The last two topics are not normally covered in elementary texts, but we have included them here for completeness. In general, all structures can be classified as either statically determinate or statically indeterminate. Part II describes techniques for dealing with statically indeterminate structures. Part III describes how the methodologies presented in Parts I and II are applied to "engineer" various types of bridges and buildings. This section is intended to identify the key issues involved in structural engineering practice. 1 Introduction to Structural Engineering Abstract A structure is an assemblage of components which are connected in such a way that the structure can withstand the action of loads that are applied to it. These loads may be due to gravity, wind, ground shaking, impact, temperature, or other environmental sources. Examples of structures employed in civil infrastructure are buildings, bridges, dams, tunnels, storage tanks, and transmission line towers. Non-civil applications include aerospace structures such as airplane fuselages, missiles; naval structures such as ships, offshore platforms; and automotive structures such as cars and trucks. Structural engineering is the discipline which is concerned with identifying the loads that a structure may experience over its expected life, determining a suitable arrangement of structural members, selecting the material and dimensions of the members, defining the assembly process, and lastly monitoring the structure as it is being assembled and possibly also over its life. In this chapter, we describe first the various types of structures. Each structure is categorized according to its particular function and the configuration of its components. We then discuss the critical issues that a structural engineer needs to address when designing or assessing the adequacy of a structure. The most important issue is preventing failure, especially a sudden catastrophic failure. We describe various failure modes: initial instability, material failure, and buckling of individual structural components. In order to carry out a structural design, one needs to specify the loading which is also a critical concern. Fortunately, the technical literature contains considerable information about loadings. We present here an overview of the nature of the different loads and establish their relative importance for the most common civil structures. Conventional structural design philosophy and the different approaches for implementing this design strategy are described next. Lastly, we briefly discuss some basic analytical methods of structural engineering and describe how they are applied to analyze structures. J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_1 3 4 1.1 1 Introduction to Structural Engineering Types of Structures and Structural Components Structures are everywhere in the built environment. Buildings, bridges, tunnels, storage tanks, and transmission lines are examples of a "structure." Structures differ in their makeup, i.e., the type and configuration of the components, and also in their function. Our approach to describing a structure is based on identifying a set of attributes which relate to these properties. 1.1.1 Structural Components The components are the basic building blocks of a structure. We refer to them as structural elements. Elements are classified into two categories according to their geometry [1]: 1. Line Elements—The geometry is essentially one-dimensional, i.e., one dimension is large with respect to the other two dimensions. Examples are cables, beams, columns, and arches. Another term for a line element is member. 2. Surface Elements—One dimension is small in comparison to the other two dimensions. The elements are plate-like. Examples are flat plates, curved plates, and shells such as spherical, cylindrical, and hyperbolic paraboloids. 1.1.2 Types of Structures A structure is classified according to its function and the type of elements used to make up the structure. Typical structures and their corresponding functions are listed in Table 1.1 and illustrated in Fig. 1.1. A classification according to makeup is listed in Table 1.2 and illustrated in Fig. 1.2. 1.2 Critical Concerns of Structural Engineering Of primary concern to a structural engineer is ensuring that the structure will not collapse when subjected to its design loading. This requires firstly that the engineer properly identify the extreme loading that the structure may experience over its design life and secondly, ensure that the forces generated internally within the structure due to external loading satisfy the conditions for force equilibrium. In general, a structure will deform, i.e., change its shape, when loaded. It may also Table 1.1 Structures classified by function Structural type Building Bridge Tunnel Tower Retaining walls Containments Platforms Function Provide shelter above ground Provide means of traversing above ground over a site Provide means of traversing underground through a site Support transmission lines and broadcasting devices Retain earth or other material Provide means of storage of materials, also enclose dangerous devices such as nuclear reactors Provide a platform for storage of materials and machinery either onshore or offshore 1.2 Critical Concerns of Structural Engineering 5 Fig. 1.1 Examples of typical structures classified by function move as a rigid body if not properly restrained. Certain structures such as airplanes and automobiles are designed to move. However, civil structures are generally limited to small motion due to deformation, and rigid body motion is prohibited. Identifying the design loads is discussed later in this chapter. We focus here on the force equilibrium requirement for civil structures. 1.2.1 Reactions Civil structures are connected to the ground at certain points called supports. When the external loading is applied to the structure, the supports develop forces which oppose the tendency of the 6 1 Introduction to Structural Engineering Fig. 1.1 (continued) structure to move. These forces are called reactions [2]. The nature and number of reactions depends on the type of support. Figure 1.3 shows the most common types of idealized structural supports for any planar structure. A roller support allows motion in the longitudinal direction but not in the transverse direction. A hinge prevents motion in both the longitudinal and transverse directions but allows rotation about the pin connection. Lastly, the clamped (fixed) support restrains rotation as well 1.2 Critical Concerns of Structural Engineering 7 Table 1.2 Structures classified by makeup Structural type Frame Truss Girder/beam Arch Cable Shell Composition • Composed of members rigidly or semirigidly connected in rectangular or triangular patterns • May be contained in a single plane (plane frame, plane grid), or in a 3D configuration (space frame) • A type of framed structure where the members are connected together at their ends with frictionless pins (plane or space truss) • Composed of straight members connected sequentially (end to end) • An additional descriptor related to the type of member cross section is used Examples are plate girders, box girders, and tub girders • Curved beams (usually in one plane) • Composed of cables and possibly other types of elements such as girders Examples are cable-stayed bridges and tensioned grids • Composed of surface elements and possibly also line elements such as beams The elements may be flat (plate structures) or curved (spherical or cylindrical roof structures) as translation with two reaction forces and one moment. Three-dimensional supports are similar in nature. There is an increase from 2 to 3 and from 3 to 6 in the number of reactions for the 3D hinge and a clamped support. 1.2.2 Initial Stability If either the number or nature of the reactions is insufficient to satisfy the equilibrium conditions, the structure is said to be initially unstable. Figure 1.4a illustrates this case. The structure consists of a triangular arrangement of members that are pinned at their ends. This combination of members forms a rigid body. However, the arrangement is supported on two roller supports which offer no resistance to horizontal motion, and consequently the structure is initially unstable. This situation can be corrected by changing one of the roller supports to a hinge support, as shown in Fig. 1.4b. In general, a rigid body is initially stable when translational and rotational motions are prevented in three mutually orthogonal directions. Even when the structure is adequately supported, it still may be initially unstable if the members are not properly connected together to provide sufficient internal forces to resist the applied external forces. Consider the four member pin-connected planar structure shown in Fig. 1.5a. The horizontal force, P, cannot be transmitted to the support since the force in member 1-2 is vertical and therefore cannot have a horizontal component. Adding a diagonal member, either 1-3 or 2-4, would make the structure stable. In summary, initial instability can occur either due to a lack of appropriate supports or to an inadequate arrangement of members. The test for initial instability is whether there are sufficient reactions and internal member forces to equilibrate the applied external loads. Assuming the structure is initially stable, there still may be a problem if certain structural components fail under the action of the extreme loading and cause the structure to lose its ability to carry load. In what follows, we discuss various failure scenarios for structures which are loaded. 8 Fig. 1.2 Structures classified by makeups 1 Introduction to Structural Engineering 1.2 Critical Concerns of Structural Engineering Fig. 1.2 (continued) 9 10 1 Introduction to Structural Engineering Fig. 1.3 Typical supports for planar structures Fig. 1.4 Examples of unstable and stable support conditions—planar structure 1.2 Critical Concerns of Structural Engineering 11 Fig. 1.5 Stabilizing an initially unstable planar structure Fig. 1.6 Stress–strain behavior of brittle and ductile materials 1.2.3 Loss of Stability Due to Material Failure In the first scenario, the level of stress in a component reaches the ultimate stress for the material, causing a material failure, which, in turn, triggers a failure of the component. This type of failure depends on the stress–strain relationship for the material. Figure 1.6 illustrates the tensile stress–extensional strain response of tension specimens fabricated from two different types of materials [3, 4]. The behavior of the first material is essentially linear up to a peak stress, σ f, at which point the material fractures and loses its ability to carry any load. This behavior is referred to as brittle behavior and obviously is not desirable from a structural behavior perspective. The second response is completely different. The initial behavior is linear up to a certain stress value defined as the yield stress, σ y. For further straining, the stress remains essentially constant. Eventually, the material stiffens and ultimately fails at a level of strain which is considerably greater than the yield strain, εy. This behavior is typical for ductile materials such as the steels used in civil structures. In practice, the maximum allowable strain is limited to a multiple of the yield strain. This factor is called the ductility ratio (μ) and is on the order of 5. Ductile behavior is obviously more desirable since a member fabricated out of a ductile material does not lose its load capacity when yielding occurs. However, it cannot carry additional loading after yielding since the resistance remains constant. From a design perspective, the structural engineer must avoid brittle behavior since it can result in sudden catastrophic failure. Ductile behavior and the associated inelastic deformation are acceptable provided that the ductility demand is within the design limit. Limit state design is a paradigm for dimensioning structural components that assumes the component is at its limit deformation state and calculates the force capacity based on the yield stress [5]. This topic is dealt with in Chap. 16. 12 1 Introduction to Structural Engineering Fig. 1.7 Behavior of a flexible member 1.2.4 Buckling Failure Mode Another possible failure scenario for a structural component is buckling. Buckling is a phenomenon associated with long slender members subjected to compressive loading [3, 4]. We illustrate this behavior using the member shown in Fig. 1.7a. As the axial loading is increased, the member remains straight until a critical load value is reached. At this point, the member adopts a deflected configuration (Fig. 1.7b) with the load remaining constant. The member force remains essentially constant as the end deflection, e, is increased (Fig. 1.7c). This load deflection behavior is similar to inelastic action in the sense that the member experiences a large deflection with essentially no increase in load. For flexible members, the critical load for buckling (Pcr) is generally less than the axial compressive strength based on yielding, therefore buckling usually controls the design. 1.2.5 Priorities for Stability Finally, summarizing and prioritizing the different concerns discussed in the previous sections, the highest priority is ensuring the structure is initially stable. If not stable, the structure will fail under an infinitesimal load. The second priority is avoiding buckling of the members. Buckling can result in large deformation and significant loss in load capacity for a member, which could cause the structure to lose its ability to support the applied loading. The third priority is limiting inelastic deformation of members under the extreme design loading. Although there is no loss in load capacity, the member cannot provide any additional load capacity, and therefore the deformation will increase significantly when the external loading is increased. We discuss this topic further in Sect. 1.4 where we present design philosophies. 1.3 Types of Loads As described above, structures must be proportioned so that they will not fail or deform excessively under the loads they may be subjected to over their expected life. Therefore, it is critical that the nature and magnitude of the loads they may experience be accurately defined. Usually, there are a number of different loads, and the question as to which loads may occur simultaneously needs to be addressed when specifying the design loading. In general, the structural engineer works with codes, which specify design loadings for various types of structures. General building codes such as the "International Building Code" [6] specify the requirements of governmental agencies for minimum 1.3 Types of Loads 13 design loads for structures and minimum standards for construction. Professional technical societies such as the American Society of Civil Engineers (ASCE) [7], the American Concrete Institute (ACI) [8], the American Institute of Steel Construction (AISC) [9], and the British Standards Institute (BSI) [10] publish detailed technical standards that are also used to establish design loads and structural performance requirements. In what follows, we present an overview of the nature of the different loads and provide a sense of their relative importance for the most common civil structures. 1.3.1 Source of Loads Loads are caused by various actions: the interaction of the structure with the natural environment; carrying out the function they are expected to perform; construction of the structure; and terrorist activities. 1.3.1.1 Interaction with the Environment Interaction with the natural environment generates the following types of loads: • • • • • • • • Gravity—gravitational force associated with mass Snow—gravity-type loading Wind—steady flow, gusts Earthquake—ground shaking resulting from a seismic event Water—scour, hydrostatic pressure, wave impact Ice—scour, impact Earth pressure—soil–structure interaction for foundations and underground structures Thermal—seasonal temperature variations The relative importance of these sources depends on the nature of the structure and the geographical location of the site. For example, building design is generally governed by gravity, snow, wind, and possibly earthquake loads. Low-rise buildings in arctic regions tend to be governed by snow loading. Underground basement structures and tunnels are designed for earth pressure, hydrostatic pressure, and possibly earthquake loads. Gravity is the dominant source of load for bridge structures. Wave and ice action control the design of offshore platforms in coastal arctic waters such as the coasts of Alaska and Newfoundland. Structures located in California need to be designed for high seismic load. Structures located in Florida need to be designed for high wind load due to hurricanes. Thermal loads occur when structural elements are exposed to temperature change and are not allowed to expand or contract. 1.3.1.2 Function Function-related loads are structure specific. For bridges, vehicular traffic consisting of cars, trucks, and trains generates gravity-type load, in addition to the self-weight load. Office buildings are intended to provide shelter for people and office equipment. A uniformly distributed gravity floor load is specified according to the nature of the occupancy of the building. Legal offices and libraries tend to have a larger design floor loading since they normally have more storage files than a normal office. Containment structures usually store materials such as liquids and granular solids. The associated loading is a distributed internal pressure which may vary over the height of the structure. 14 1 Introduction to Structural Engineering 1.3.1.3 Construction Construction loading depends on the process followed to assemble the structure. Detailed force analyses at various stages of the construction are required for complex structures such as segmented long-span bridges for which the erection loading dominates the design. The structural engineer is responsible for approving the construction loads when separate firms carry out engineering and construction. A present trend is for a single organization to carry out both the engineering design and construction (the design-build paradigm where engineering companies and construction companies form a joint venture for the specific project). In this case, a team consisting of structural engineers and construction engineers jointly carries out the design. An example of this type of partnering is the construction of the Millau Viaduct in southwestern France, shown in Fig. 1.8. The spans were constructed by cantilevering segments out from existing piers, a technically challenging operation that required constant monitoring. The bridge piers are the highest in the world: the central pier is 280 m high. 1.3.1.4 Terrorist Loads Terrorist loads are a new problem for structural engineers, driven primarily by the need to protect essential facilities from terrorist groups. Design criteria are continuously evolving, and tend to be directed more at providing multilevel defense barriers to prevent incidents, rather than to design for a specific incident. Clearly, there are certain incidents that a structure cannot be designed to safely handle, such as the plane impacts that destroyed the World Trade Center Towers. Examining progressive collapse mechanisms is now required for significant buildings and is the responsibility of the structural engineer. Fig. 1.8 Millau Viaduct 1.3 Types of Loads 1.3.2 15 Properties of Loadings The previous discussion was focused on the source of loadings, i.e., environmental, functional, construction, and terrorist activity. Loadings are also characterized by attributes, which relate to properties of the loads. Table 1.3 lists the most relevant attributes and their possible values. Duration relates to the time period over which the loading is applied. Long-term loads, such as self-weight are referred to as dead loads. Loads whose magnitude or location changes are called temporary loads. Examples of temporary loads are the weight of vehicles crossing a bridge, stored items in buildings, wind and seismic loads, and construction loads. Most loads are represented as being applied over a finite area. For example, a line of trucks is represented with an equivalent uniformly distributed load. However, there are cases where the loaded area is small, and it is more convenient to treat the load as being concentrated at a particular point. A member partially supported by cables such as a cable-stayed girder is an example of concentrated loading. Temporal distribution refers to the rate of change of the magnitude of the temporary loading with time. An impulsive load is characterized by a rapid increase over a very short duration and then a drop off. Figure 1.9 illustrates this case. Examples are forces due to collisions, dropped masses, brittle fracture material failures, and slamming action due to waves breaking on a structure. Cyclic loading alternates in direction (+ and) and the period may change for successive cycles. The limiting case of cyclic loading is harmonic excitation where the amplitude and period are constant. Seismic excitation is cyclic. Rotating machinery such as printing presses, electric generators, and turbines produce harmonic excitation on their supports when they are not properly balanced. Quasi-static loading is characterized by a relatively slow build up of magnitude, reaching essentially a steady state. Because they are applied slowly, there is no appreciable dynamic amplification and the structure responds as if the load was a static load. Steady winds are treated as quasi-static; wind gusts are impulsive. Wind may also produce a periodic loading resulting from vortex shedding. We discuss this phenomenon later in this section. The design life of a structure is that time period over which the structure is expected to function without any loss in operational capacity. Civil structures have long design lives vs. other structures such as motorcars, airplanes, and computers. A typical building structure can last several centuries. Bridges are exposed to more severe environmental actions, and tend to last a shorter period, say 50–75 years. The current design philosophy is to extend the useful life of bridges to at least 100 years. Table 1.3 Loading attributes Attribute Duration Spatial distribution Temporal distribution Degree of certainty Fig. 1.9 Temporal variation of loading. (a) Impulsive. (b) Cyclic. (c) Quasi-static Value Temporary or permanent Concentrated or distributed Impulsive; cyclic; quasi-static Return period; probability of occurrence 16 1 Introduction to Structural Engineering The Millau viaduct shown in Fig. 1.8 is intended to function at its full design capacity for at least 125 years. Given that the natural environment varies continuously, the structural engineer is faced with a difficult problem: the most critical natural event, such as a windstorm or an earthquake that is likely to occur during the design life of the structure located at a particular site needs to be identified. To handle this problem, natural events are modeled as stochastic processes. The data for a particular event, say wind velocity at location x, is arranged according to return period which can be interpreted as the average time interval between occurrences of the event. One speaks of the 10-year wind, the 50-year wind, the 100-year wind, etc. Government agencies have compiled this data, which is incorporated in design codes. Given the design life and the value of return period chosen for the structure, the probability of the structure experiencing the chosen event is estimated as the ratio of the design life to the return period. For example, a building with a 50-year design life has a 50 % chance of experiencing the 100-year event during its lifetime. Typical design return periods are 50 years for wind loads and between 500 and 2500 years for severe seismic loads. Specifying a loading having a higher return period reduces the probability of occurrence of that load intensity over the design life. Another strategy for establishing design loads associated with uncertain natural events is to increase the load magnitude according to the importance of the structure. Importance is related to the nature of occupancy of the structure. In ASCE Standards 7-05 [7], four occupancy categories are defined using the potential hazard to human life in the event of a failure as a basis. They are listed in Table 1.4 for reference. The factor used to increase the loading is called the importance factor, and denoted by I. Table 1.5 lists the values of I recommended by ASCE 7-05 [7] for each category and type of loading. For example, one increases the earthquake loading by 50 % for an essential structure (category 4). 1.3.3 Gravity Live Loads Gravitational loads are the dominant loads for bridges and low-rise buildings located in areas, where the seismic activity is moderate. They act in the downward vertical direction and are generally a combination of fixed (dead) and temporary (live) loads. The dead load is due to the weight of the construction materials and permanently fixed equipment incorporated into the structure. As Table 1.4

Occupancy categories Category I II III IV Description Structures that represent a low hazard to human life in the event of a failure All structures except those listed in categories I, III, and IV Structures that represent a substantial hazard to human life in the event of failure Essential structures. Failure not allowed Table 1.5 Values of I Category I II III IV Wind Non-horizontal 0.87 1.00 1.15 1.15 Horizontal 0.77 1.00 1.15 1.15 Snow 0.80 1.00 1.10 1.20 Earthquake 1.00 1.00 1.25 1.50 1.3 Types of Loads 17 Table 1.6 Uniformly distributed live loads (ASCE 7-05) Occupancy Computer equipment Dormitories File room Court rooms Scientific laboratories Public rooms Rest rooms Laundries Foundries Ice manufacturing Transformer rooms Storage, hay, or grain Magnitude lbs/ft2 (kN/m2) 150 (7.18) 80 (3.83) 125 (6.00) 50–100 (2.4–4.79) 100 (4.79) 100 (4.79) 60 (2.87) 150 (7.18) 600 (28.73) 300 (14.36) 200 (9.58) 300 (14.36) mentioned earlier, temporary live loads depend on the function of the structure. Typical values of live loads for buildings are listed in Table 1.6. A reasonable estimate of live load for office/residential facilities is 100 lbs/ft2 (4.8 kN/m2). Industrial facilities have higher live loadings, ranging up to 600 lbs/ft2 for foundries. Live loading for bridges is specified in terms of standard truck loads. In the USA, bridge loads are defined by the American Association of State Highway and Transportation Officials (AASHTO) [11]. They consist of a combination of the Design truck or tandem, and Design lane load. The design truck loading has a total weight of 72 kip (323 kN), with a variable axle spacing is shown in Fig. 1.10. The design tandem shall consist of a pair of 25 kip (112 kN) axles spaced 4 ft (1.2 m) apart. The transverse spacing of wheels shall be taken as 6 ft (1.83 m). The design lane load shall consist of a load of 0.64 kip/ft2 (30.6 kN/m2) uniformly distributed in a 10 ft (3 m) width in the transverse direction. 1.3.4 Wind Loading 1.3.4.1 Wind Pressure Distribution The effect of wind acting on a building is represented by a pressure loading distributed over the exterior surface. This pressure loading depends on the geometry of the structure and the geographic location of the site. Figure 1.11 illustrates the flow past a low-rise, single story, flat roof structure. The sharp corners such as at point A causes flow separation, resulting in eddies forming and turbulence zones on the flat roof, side faces, and leeward face. The sense of the pressure is positive (inward) on the incident face and negative (outward) in the turbulence zones. In general, the magnitude of the pressure varies over the faces, and depends on both the shape of the structure and the design wind velocity at the site. The influence of shape is illustrated by Fig. 1.12, which shows the effect of roof angle on the pressure distribution. When θ > 45 , there is a transition from negative to positive pressure on face AB of the inclined roof. This shift is due to the flow separation point moving from A to B for steeply inclined roofs. 1.3.4.2 Wind Velocity The effect of wind is characterized firstly by the topography at the site, and secondly by the regional wind environment. Exposure categories are defined to describe the local topography and the local exposure to wind. ASCE 7-05 adopts the following definitions of exposure categories. 18 Fig. 1.10 Characteristics of the AASHTO design truck Fig. 1.11 Flow lines and pressure distributions due to wind 1 Introduction to Structural Engineering 1.3 Types of Loads 19 Fig. 1.12 Wind pressure profiles for a gable roof. (a) θ < 45 . (b) θ > 45 Category B: Site located within an urban or suburban area having numerous closely spaced obstructions similar in size to a single family dwelling, and extending at least 2600 ft from the site. Category D: Site located in a flat unobstructed area or on a water surface outside hurricane prone regions, and extending at least 5000 ft from the site. Category C: All cases where exposure categories B and D do not apply. Regional wind environments are represented by maps containing wind speed data for a specified return period and exposure category. Figure 1.13 shows US data for the 50-year wind speed observed at 10 m elevation and exposure to Exposure C. The higher wind speeds along the East and Gulf Coasts reflects the occurrence of hurricanes in these regions. Typical 50-year wind speeds are on the order of 100 miles per hour (45 m/s). Given a site, one can establish the 50-year wind speed at 10 m elevation using Fig. 1.13. In general, the wind velocity increases with distance from the ground. A typical approximation is a power law: V ðzÞ ¼ V z1 =α z 0 ð1:1Þ where z is the elevation above the ground, V is the velocity measured at elevation z, and α 7. For US data, one takes z ¼ 10 m and V given by Fig. 1.13. 1.3.4.3 Pressure Profiles The next step is to establish the vertical pressure distribution associated with this velocity distribution, and then modify it to account for the shape of the building. Pressure and velocity are related by Bernoulli's Equation, which is a statement of conservation of energy. Specialized for steady irrotational inviscid flow of a weightless fluid, the Law states that [12] 1 E ¼ Energy per unit volume ¼ p þ ρV 2 2 ð1:2Þ is constant along a streamline. Here, p is the pressure, ρ is the mass density, and 1/2ρV2 is the kinetic energy per unit volume. Assuming the pressure is zero in the free stream flow regime away from the structure, and taking point (1) in the free stream and point (2) at the structure, one obtains 1 p2 ¼ ρ V 21 V 22 2 ð1:3Þ The free stream velocity, V1, is defined by (1.1). Considering the flow to be stopped by the structure, (V2 0), it follows that the maximum pressure energy associated with the free stream velocity is estimated as 20 1 Introduction to Structural Engineering Fig. 1.13 Basic wind speed miles per hour (meter per second) for the East Coast of the USA 1 pp2 |max ¼ ρV 21 ¼ pstag 2 ð1:4Þ This pressure is called the stagnation pressure and is generally expressed in terms of the reference velocity, V, at z ¼ 10 m and a function k(z) which defines the vertical distribution. 1 2 pstag ¼ ρV kðzÞ 2 2 ð1:5Þ ASCE 7-05 tabulates values of k(z) vs. z. The actual pressure distribution is influenced by the geometric shape which tends to change both the magnitude and sense of the pressure. Figures 1.11 and 1.12 illustrate this effect for flat and gable roof structures. Design codes handle this aspect by introducing "shape" factors for different regions of the structural surface. They also include a gust factor for "dynamic" loading, and an importance factor for the structure type. The final expression for the design pressure has the following general form: Pdesign ¼ IGCp ðzÞpstag ð1:6Þ where Cp(z) is the pressure coefficient that accounts for the shape, G is the gust factor, and I is the importance factor corresponding to the occupancy category. Values for these parameters are code dependent. The determination of the design pressure can be labor intensive if one wants to account 1.3 Types of Loads 21 for the spatial distribution of design pressure. A reasonable estimate of the design pressure for low-rise buildings. Example 1.1 Wind Pressure Distribution on a Low-Rise Gable Roof Structure Given: The structure shown in Fig. E1.1a. There are four surface areas included in the sketch. Zone (1) is the windward face, zone (2) is the leeward face, and zones (3) and (4) are on the gable roof. Fig. E1.1a Determine: The wind pressure distribution on the interior zone away from the ends. Assume V ¼ 100 mph and exposure C Solution: Applying (1.5) leads to pstag ¼ 0:00256 104 kðzÞ ¼ 25:6kðzÞ lb=ft2 Values of k(z) and the corresponding pstag are listed below z (ft) 15 20 25 30 k (z) 0.85 0.90 0.94 0.98 pstag (lb/ft2) 21.8 23.0 24.1 25.1 We assume the structure is Category III and use I ¼ 1.15. For low-rise buildings with h < 60 ft, the factors G and Cp are combined and specified as constant for each zone. Using data from ASCE 7-05, the values are Zone 1 2 3 4 GCp 0.53 0.43 0.69 0.48 IGCp 0.609 0.494 0.552 22 1 Introduction to Structural Engineering Fig. 1.14 Lateral bracing systems Lastly, we compute the design pressure using (1.6). The ASCE 7-05 code assumes that pdesign varies on the windward force (zone 1), but specifies constant distributions for the other zones. The details are listed below. Zone 1 Zone 2 Zone 3 Zone 4 pdesign pdesign pdesign pdesign ¼ ¼ ¼ ¼ 0.609 (25.6) k(z) ¼ 15.59 k(z) 0.495 (25.6) k(30) ¼ 12.42 psf 0.791 (25.6) k(30) ¼ 19.92 psf 0.552 (25.6) k(30) ¼ 13.85 psf Pressure distributions generated with (1.6) define the quasi-static wind loads. The wind loads act on the dominant loads and generally control the structural dimensioning process for vertical members. Since the wind loads are primarily horizontal (lateral) direction. For low-rise buildings, the dominant loads are the gravity loads, whereas the gravity loads are vertical, lateral structural bracing systems as shown in Fig. 1.14 need to be incorporated in certain types of structures such as braced frames. This topic is addressed further in Chaps. 11, 14, and 15. 1.3.4.4 Vortex Shedding Pressure The action of a steady wind on a structure is represented by quasi-static forces. However, a steady wind also creates periodic forces due to the shedding of vortices from the turbulence zones at the leeward face [12]. Consider the rectangular cross-section plan view shown in Fig. 1.15. As the incident flow velocity increases, eddies are created at the upper and lower surfaces and exit downstream. This shedding pattern develops a cyclic mode, shedding alternately between the upper and lower surfaces, which result in an antisymmetric pressure distribution. The net effect is a periodic force, Ft, acting in the transverse direction with frequency fs. An estimate of the shedding frequency is f s 0.2V/D ð1:7Þ where D is a representative dimension in the transverse direction, and V is the free stream velocity. Vortex shedding is a major concern for tall buildings and slender long-span horizontal structures since these structures are flexible and consequently more susceptible to transverse periodic excitation with frequency close to fs. Low-rise buildings are stiffer and relatively insensitive to vortex shedding-induced transverse motion. 1.3.5 Snow Loading Design snow loads for a structure are based on ground snow load data for the region where the structure is located. Snow loads act on the roof zones of structures. For a flat roof, defined as a roof with a slope angle less than 5 , the snow load is represented as a uniformly downward pressure, pf. The 1.3 Types of Loads 23 Fig. 1.15 Vortex shedding patterns—plan view magnitude of pf depends on the exposure category and regional environment at the site, as well as the importance of the structure. We express pf as pf ¼ Cpg ð1:8Þ where pg is the ground snow pressure given by Fig. 1.16a and C is a factor that incorporates the exposure and importance parameters. A typical value of C is 1. The ground snow pressure varies from 0 in the southeastern zone of the USA up to 100 psf in northern New England. A sloped roof is defined as a roof with a slope angle greater than 5 . The snow load on a sloped roof is expressed in terms of the horizontal projected area rather than the actual surface area. Figure 1.16b illustrates this definition. The sloped roof pressure depends on the slope angle and is defined as ps ¼ CS pf ð1:9Þ where Cs is a slope coefficient. In general, Cs 1. For θ ≤ 30 , one usually assumes Cs 1 and takes ps pf. When the roof has projections as illustrated in Fig. 1.17, a nonuniform snow pressure loading can result due to the drifting on both the windward and leeward faces caused by wind. Drifts are modeled as triangular surcharge loadings. The details are code dependent. 1.3.6 Earthquake Loading The structural engineer's task is to design structures such that they can resist the ground shaking associated with an earthquake without collapsing. Since an earthquake may occur anytime during the design life, the first task is to identify the magnitude of peak ground acceleration (ρga) that has a specified probability of occurrence during the design life. A common value is 2 % probability of occurrence in 50 years, which corresponds to a return period of 2500 years. Earthquake ground motion is site specific in that it depends on the location and soil conditions for the site. Sites near known faults and sites on soft soils such as soft clay experience more intense ground motion. Factors such as the importance of the building, the geographic location, the site, and the type of soil must be taken into account when specifying the design magnitude for ρga. In order to understand how buildings respond to ground motion, one needs to examine the dynamic response. Consider the three-story frame shown in Fig. 1.18a. We approximate it with the single beam/mass system defined in Fig. 1.18b. This approximation, known as a single degree-of-freedom 24 1 Introduction to Structural Engineering Fig. 1.16 Snow loadings on sloped and flat roofs. (a) Flat roof. (b) Sloped roof Fig. 1.17 Snow drift profiles 1.3 Types of Loads 25 Fig. 1.18 A typical three-story frame and the corresponding one degree-of-freedom model Fig. 1.19 Peak lateral inertia force model, provides useful information concerning the influence of certain structural properties on the response. The ground acceleration is defined as ag. This motion causes the mass to vibrate. We define aT,max as the peak total acceleration of the mass (Fig. 1.19). If the frame is very stiff, aT,max is essentially equal to ag,max, the peak ground acceleration. Conversely, if the frame is very flexible, aT,max is small in comparison to ag,max. It follows that the stiffness of the structure has a significant influence on the peak total acceleration response. The peak acceleration also depends on the geographic location and the soil conditions at the site. Data concerning earthquake accelerations is published by the US Geological Survey on their Web site [13]. This site contains an extensive set of earthquake ground motion records for the USA and other major seismically active regions throughout the world. The motion of the mass generates an inertia force which is resisted by the lateral shear force in the system. The maximum value of the lateral shear force is denoted as Vmax. 26 1 Introduction to Structural Engineering Fig. 1.20 Seismic lateral load profile V max ¼ MaT, max ¼ W aT, max g ð1:10Þ Given the structural weight, W, and the peak total acceleration, one can estimate the peak total lateral load that the structure will experience due to seismic excitation. This load is assumed to be distributed linearly throughout the height of the structure, as indicated in Fig. 1.20, and is used to generate an initial structural design. The final design is checked with a more refined dynamic analysis. Seismic design is an advanced topic within the field of structural engineering. We discuss this topic in more detail in Chap. 14. 1.4 Structural Design Philosophy Conventional structural design philosophy is based on satisfying two requirements, namely safety and serviceability [7]. Safety relates to extreme loadings, which have a very low probability of occurrence, on the order of 2 %, during a structure's life, and is concerned with the collapse of the structure, major damage to the structure and its contents, and loss of life. The most important priority is ensuring sufficient structural integrity so that sudden collapse is avoided. Serviceability pertains to medium to large loadings, which are likely to occur during the structure's lifetime. For service loadings, the structure should remain operational. It should suffer minimal damage, and furthermore, the motion experienced by the structure should not exceed specified comfort levels for humans and motion-sensitive equipment mounted on the structure. Typical occurrence probabilities for service loads range from 10 to 50 %. Safety concerns are satisfied by requiring the resistance, i.e., the strength of the individual structural elements to be greater than the demand associated with the extreme loading. Once the structure is dimensioned, the stiffness properties are derived and used to check the various serviceability constraints such as elastic behavior. Iteration is usually necessary for convergence to an acceptable structural design. This approach is referred to as strength-based design since the elements are dimensioned initially according to strength requirements. Applying a strength-based approach for preliminary design is appropriate when strength is the dominant design requirement. In the past, most structural design problems have fallen in this 1.5 Basic Analytical Tools of Structural Analysis 27 category. However, the following developments have occurred recently that limit the effectiveness of the strength-based approach. Firstly, the trend toward more flexible structures such as tall buildings and long-span horizontal structures has resulted in more structural motion under service loading, thus shifting the emphasis toward serviceability. Secondly, some new types of facilities such as micro device manufacturing centers and hospital operating centers have more severe design constraints on motion than the typical civil structure. For example, the environment for micro device manufacturing must be essentially motion free. Thirdly, recent advances in material science and engineering have resulted in significant increases in the strength of traditional civil engineering materials. However, the material stiffness has not increased at the same rate. The lag in material stiffness vs. material strength has led to a problem with satisfying the requirements on the various motion parameters. Indeed, for very high strength materials, the motion requirements control the design. Fourthly, experience with recent earthquakes has shown that the cost of repairing structural damage due to inelastic deformation is considerably greater than anticipated. This finding has resulted in a trend toward decreasing the reliance on inelastic deformation to dissipate energy and shifting to other type of energy dissipating and energy absorption mechanisms. Performance-based design [14] is an alternate design paradigm that addresses these issues. The approach takes as its primary objective the satisfaction of motion-related design requirements such as restrictions on displacement and acceleration and seeks the optimal deployment of material stiffness and motion control devices to achieve these design targets as well as satisfy the constraints on strength. Limit state design can be interpreted as a form of performance-based design where the structure is allowed to experience a specific amount of inelastic deformation under the extreme loading. 1.5 Basic Analytical Tools of Structural Analysis Analyzing a structure involves not only dimensioning the structure but also evaluating whether the structure's response under the construction and design loadings satisfy the specified design criteria. Response evaluation is commonly referred to as structural analysis and is carried out with certain analytical methods developed in the field of Engineering Mechanics and adopted by structural engineers. In this section, we review these methods and illustrate their application to some simple structures. Most of this material is covered in textbooks dealing with Statics and Mechanics of Materials [2–4] and Structural Analysis [15–17]. Heyman's text [18] contains an excellent description of the "underlying science of Structural Engineering." 1.5.1 Concept of Equilibrium: Concurrent Force System We begin with a discussion of static equilibrium conditions for solid bodies. This topic is relevant to structural engineering since structures are solid bodies subjected to loads, and we need to ensure that a structure remain at rest, i.e., that it is in a state of equilibrium. The simplest case is a body subjected to a set of concurrent forces. By definition, the lines of action of the forces comprising a concurrent force system intersect at a common point. Figure 1.21 illustrates this case. For static equilibrium, the resultant force of the force system must be a null vector. R ¼ F1 þ F2 þ F3 ¼ 0 ð1:11Þ 28 1 Introduction to Structural Engineering Fig. 1.21 Concurrent force system We convert the vector equilibrium over to a set of algebraic equations by resolving the force vectors into their components with respect to an arbitrary set of orthogonal directions (X, Y, Z). This operation leads to 3 X Fi, x ¼ F1, x þ F2, x þ F3, x ¼ 0 i¼1 3 X Fi, Y ¼ F1, Y þ F2, Y þ F3, Y ¼ 0 i¼1 3 X Fi, z ¼ F1, z þ F2, z þ F3, z ¼ 0 i¼1 ð1:12Þ We find it more convenient to work with (1.12) rather than (1.11). When all the force vectors are in one plane, say the X Y plane, the force system is called a planar force system and (1.12) reduces to two equations. Most of the force systems that we deal with will be planar systems. 1.5.2 Concept of Equilibrium: Nonconcurrent Force System The next level of complexity is a body subjected to a nonconcurrent planar force system. Referring to Fig. 1.22, we note the forces tend to rotate the body as well as translate it. The static equilibrium requirement is that the resultant force vector to vanish and, in addition, the resultant moment vector about an arbitrary point must also vanish. R ¼ F1 þ F2 þ F3 ¼ 0 Mo ¼ 0 ð1:13Þ Resolving the force and moment vectors into X, Y, Z components leads to six scalar equations, three for force and three for moment. When the force system is planar, say in the X Y plane, (1.13) reduce to three scalar equations 1.5 Basic Analytical Tools of Structural Analysis 29 Fig. 1.22 Nonconcurrent force system X Fi, X ¼ 0 X Fi, Y ¼ 0 ð1:14Þ X Mo ¼ 0 where o is an arbitrary point in the x y plane. Note that now for a planar system there are three equilibrium conditions vs. two for a concurrent system. Note also that these conditions involve only the external forces applied to the body. Example 1.2 Equilibrium Equations Given: The rigid body and force system shown in Fig. E1.2a. Forces AX, AY, and BY are unknown. Fig. E1.2a Determine: The forces AX, AY, and BY 30 1 Introduction to Structural Engineering Solution: We sum moments about A, and solve for BY X MA ¼ 40ð1:5Þ 20ð2Þ þ BY ð5Þ ¼ 0 BY ¼ þ4 kN " Next, we sum the forces in the X and Y directions leading to (Fig. E1.2b) X Fx ¼ Ax þ 20 ¼ 0) Ax ¼ 200 Ax ¼ 20 kN ← Fy ¼ AY þ 4 þ 20 ¼ 0) AY ¼ 52) AY ¼ 52 kN # Fig. E1.2b 1.5.3 Idealized Structure: Free Body Diagrams Generating an idealization of an actual structure is the key step in applying the equilibrium equations. Given a structure acted upon by external loads and constrained against motion by supports, one idealizes the structure by identifying the external loads and supports, and replacing the supports with their corresponding unknown reaction forces. This process is called constructing the free body diagram (FBD). Figure 1.23a,b illustrates the details involved. One applies the equilibrium equations to the FBD. Note that all the forces acting on the diagram have four unknown reaction forces. Since there are only three equilibrium equations, one cannot determine all the reaction forces using only the equilibrium equations. In this case, we say that the structure is statically indeterminate. Constructing an FBD is an essential step in applying the equilibrium equations. The process is particularly useful when the structure is actually a collection of interconnected structural components such as a framed structure. One first generates an FBD for the entire structure and then works with separate FBDs for the individual members. We illustrate this approach throughout the text. 1.5.4 Internal Forces Consider the body shown in Fig. 1.24a. Suppose we pass a cutting plane as indicated and separate the ~ From two segments. We represent the action of body "n" on body "m" by a force ~ F and moment M. Newton's third law, the action of body m on body n is opposite in sense. ~ by applying the equilibrium Once the reaction forces are known, we can determine ~ F and M conditions to either segment. These force quantities are called "internal forces" in contrast to the 1.5 Basic Analytical Tools of Structural Analysis 31 Fig. 1.23 Constructing the free body diagram (FBD) Fig. 1.24 Definition of internal forces which are "external forces." Note that the magnitude of the internal forces varies with the location of the cutting plane. The following example illustrates the process of computing internal forces. 32 1 Introduction to Structural Engineering Example 1.3 Given: The body and loading shown in Fig. E1.3a. Fig. E1.3a Determine: The internal forces at Sects. 1-1 and 2-2 Solution: First, we determine the reactions at A and B by applying the equilibrium conditions to the entire body AB (Fig. E1.3b). Fig. E1.3b The static equilibrium equations are X Fx ¼ 0 X Fy ¼ 0 X Mabout A ¼ 0 Ax ¼ 0 AY þ BY ¼ 30 8ð20Þþ 16ð10Þ 24BY ¼ 0 2 We solve for BY, BY ¼ 20 3 þ ð10Þ ¼ 13:33 kip " and then AY AY ¼ 16:7 kip " 1.5 Basic Analytical Tools of Structural Analysis 33 Next, we work with the FBDs shown below. We replace the internal force vector with its normal and tangential components, F and V (Figs. E1.3c and E1.3d). Fig. E1.3c Left segment-cutting plane 1-1 Applying the equilibrium conditions to the segment shown below leads to P Fx ¼P F ¼ 0 Fy ¼ 16:7 V 1 ¼ 0) V 1 ¼ 16:7 kip # P Mabout 1 ¼ 0 M11 16:7ð6Þ ¼ 0) M11 ¼ 100:2 kip ft counterclockwise Applying the equilibrium conditions to the segment shown below leads to P Fx ¼ 0 P Fy ¼ 16:7 V 2 ¼ 0) V 2 ¼ 3:3 kip # M22 ¼ 122ð16:7Þ þ 20 8ð10Þ ¼ 3:3 kip ft counterclockwise Fig. E1.3d Left segment-cutting plane 2-2 Note that the values of V11 and V22 are opposite from those we chose initially. 1.5.5 Deformations and Displacements When a body is subjected to external loads, it responds by developing internal forces which are needed to maintain equilibrium between the external forces and the internal forces. These internal forces produce stresses which cause the body to change its shape and displace from its unloaded position. 34 1 Introduction to Structural Engineering Fig. 1.25 Unformed and deformed states Consider the member shown in Fig. 1.25. We apply an axial force which generates the axial stress, σ, equal to σ¼ F A ð1:15Þ where A is the cross-sectional area. The resulting strain depends on E, the modulus of elasticity for the material [3, 4]. ε¼ σ E ð1:16Þ By definition, the extensional strain is the relative change in length. ε¼ ΔL L ð1:17Þ Then, ΔL ¼ Lε ¼ L F AE ð1:18Þ We refer to the movement due to strain as the displacement and denote it by u. It follows that ΔL is equal to u. Finally, we write (1.18) as u¼ L u¼ F ð1:19Þ AE Strains are generally referred to as deformations since they refer to a change in shape. In this example we illustrate that displacements are a consequence of deformations that are due to the forces. We refer to deformations and displacements as deformations. Note that deformations are dimensionless quantities whereas displacements have geometric units such as either length (translation) or angle (rotation). The coefficient of F in (1.19) has units of displacement/force. We interpret this coefficient as a measure of the flexibility of the member. Here, we are defining flexibility as displacement per unit force. The inverse of flexibility is called stiffness. Stiffness relates the force required to introduce a unit displacement. Inverting (1.19) leads to AE F¼ u ð1:20Þ L It follows that the stiffness of an axial loaded member is equal to AE L. Stiffness and flexibility are important concepts in structural engineering. We use them to reason qualitatively about the change in behavior of a structure when we introduce modifications to the geometry and structural members. Obviously, to reduce displacements, one needs to make the structure stiffer. How this is achieved is one of the themes of this text. 1.5 Basic Analytical Tools of Structural Analysis 1.5.6 35 Structural Behavior: Structural Analysis When a structure is subjected to an external loading, it responds by developing internal forces which lead to internal stresses. The stresses generate strains, resulting in displacements from the initial unloaded position. Figure 1.26 illustrates the displacement process for a beam-type member subjected to a transverse loading. This process continues until the internal stresses reach a level at which the external loading is equilibrated by the internal forces. The final displacement profile corresponds to this equilibrium state. Structural analysis is concerned with quantifying the response of structures subjected to external loading. The scope includes determining the magnitude of the reactions, internal forces, and displacements. The analysis is generally carried out in the order shown in Fig. 1.26. Fig. 1.26 Simple beam response 36 1 Introduction to Structural Engineering 1.5.6.1 Study Forces In the study of forces, we apply the equilibrium equations to various FBDs. We work initially with the FBD for the structure treated as a single body and determine the reactions. Once the reactions are known, we select various cutting planes and determine the corresponding internal forces. This phase involves some heuristic knowledge as to the "best" choice of cutting planes. 1.5.6.2 Study Displacements Displacements are the geometric measures that define the position of the structure under the applied external loading. Displacements are a consequence of internal stresses and are usually expressed in terms of the internal forces. The form of the "force-displacement" relations depends on the type of structural member, e.g., a truss member and a beam. We discuss this topic in more detail in Chaps. 2 and 3. In what follows, we illustrate these computations for some fairly simple structures. Example 1.4 Given: The structure defined in Fig. E1.4a. Member AB is a rigid member. It is connected to a hinge support at A, and supported at B by a cable, BC. Determine: The reactions, cable tension, and vertical displacement at B. Assume EC ¼ 200 GPa, AC ¼ 600 mm2, h ¼ 4 m, L ¼ 10 m, and P ¼ 80 kN Fig. E1.4a Solution: We start with the FBD of the entire structure shown in Fig. E1.4b. We note that the cable force is tension. Requiring the sum of the moments of the forces with respect to point A to vanish leads to the tension T 1.5 Basic Analytical Tools of Structural Analysis 37 Fig. E1.4b X L Mabout A ¼ P LT C ¼ 0 2 þ TC ¼ P 2 Next, we determine the reactions at A using force summations. X X FY ¼ 0 Ay þ T C P ¼ 0) Ay ¼ Fx ¼ 0 Ax ¼ 0 P 2 The vertical displacement of B is equal to the extension of the cable. Noting (1.19), the expression for vB is h h P 4; 000 80 vB ¼ ¼ ¼ 1:33 mm # TC ¼ A c E c A c E C 2 ð600Þ200 2 In what follows, we illustrate the application of the general analysis procedure to the idealized structure defined in Fig. 1.27. Member ABCD is considered to be rigid. It is supported by a hinge at A and springs at C and D. The force, P, is constant. Replacing the hinge support and springs with their corresponding forces results in the FBD shown in Fig. 1.27b. There are four unknown forces; AX, AY, Fc, and Fd. Setting the resultant force equal to zero leads to AX ¼ 0 Ay þ Fc þ Fd ¼ P Summing moments about A, we require that the moment vanish at A. 1 1 P ¼ Fc þ Fd 4 2 ð1:21Þ Since there are only two equations and three unknowns, the structure is statically indeterminate. 38 1 Introduction to Structural Engineering Fig. 1.27 Rigid member on springs Fig. 1.28 Deformation modes. (a) Extension. (b) Shear We generate additional equations by examining how the structure deforms. Deformation is a consequence of applying a force to a material. Deformation is measured with a change in shape. Figure 1.28 illustrates various deformation modes: the first is extension of a spring; the second is shear. A rigid body is an idealized case: the deformations are considered to be negligible. An important phase in the analysis of a deformable body is the study of deformations. One first identifies the displacement variables that define the "deformed" position and then, using geometric analysis, establishes the expressions relating the deformations of the deformable structural elements with the displacements. We illustrate this process for the structure defined in Fig. 1.29. 1.5 Basic Analytical Tools of Structural Analysis 39 Fig. 1.29 Deformation– displacement relations Member ABCD is assumed to be rigid and therefore it does not deform. Deformation occurs in the springs at C and D, causing ABCD to rotate about the hinge at A. There is only one independent displacement variable. We take θ to be the rotation angle shown in Fig. 1.29. With this choice of sense, the springs compress. When θ is small, the spring deformations can be approximated as linear functions of θ. This approximation is valid for most cases. 1 ec ¼ θ 2 ð1:23Þ ed ¼ 1θ The last step in the analysis involves relating the deformations and the corresponding internal forces. For this example structure, the internal forces are the spring forces, Fc and Fd. The relationship between the force and deformation of a component is a function of the makeup of the component (i.e., the material used and the geometry of the component). Here, we assume the behavior is linear and write Fc ¼ k c e c Fd ¼ k d e d ð1:24Þ where kc and kd are the spring stiffness factors. Note that the units of k are force/length since e has units of length. At this point, we have completed the formulation phase. There are seven equations, (1.21)–(1.23), relating the seven variables consisting of the four forces, one displacement, and two deformations. Substituting for ec and ed in (1.24) with (1.23), we observe that the reaction Av can be determined once the spring forces are known. Therefore, we need to solve (1.24) and find the spring forces. We now proceed through the various steps. 1.5.6.3 Solution Procedure We reduce the problem to solving for θ using the equilibrium equation in reverse, and then solving the remaining equation. We use (1.23) and (1.24) to solve for the deformations. The resulting equations together with (1.24) are 1 Fc ¼ kc θ, Fd ¼ kd θ ð1:25Þ 2 1 1 P Fc þ Fd 4 2 The most convenient strategy is to substitute for Fc, Fd in the second equation. 1 1 P ¼ kc þ kd θ 4 4 ð1:25bÞ 40 1 Introduction to Structural Engineering θ¼ P δ4kd þ kc Þ ð1:26Þ Finally, the spring forces corresponding to this value of θ are Fc ¼ kc P 2δ4kd þ kc Þ P Fd ¼ kd P δ4kd þ kc Þ ð1:27Þ An alternate strategy is to solve first for one of the spring forces. Suppose we take Fc as the primary force variable. Using (1.25b), we solve for Fd. 1 Fc ¼ 1 Fd ¼ P ¼ Fd 2 ð1:28Þ Another equation is required relating Fc and Fd. This is obtained by eliminating θ in (1.25a). The steps are θ¼ 1 Fc kc δ1=2Þ ¼ 1 Fd kd Fd ¼ 2kd Fc kc ð1:30Þ and Fd ¼ Equation (1.30) represents a constraint on the spring forces. The deformations of the springs are not arbitrary but they must satisfy (1.23), which can be written as: ed ¼ 2ec Finally, substituting for Fd in (1.28) and solving for Fc leads to 1 ¼ 2 Fc ¼ Fc 1 P þ 4kd =kc ð1:31Þ ð1:32Þ The rotation angle is determined with (1.29) and Fc with (1.30). We refer to the first solution procedure as the displacement or stiffness method. The displacement method is the most commonly used solution scheme. Most structural analysis computer programs are based on the displacement method. The second procedure is called the force or flexibility method. Some manipulation of the equations is required to establish the equation for the force. In general, it is somewhat more difficult to apply in this case. However, the Force Method is more convenient to apply than the displacement method when the structure is statically determinate, since the forces can be determined using only the equilibrium equations. The approach taken in part I of the text is based on the Force Method. Later in part II, we discuss the Force and Displacement methods in more detail. 1.5.7 The Importance of Displacements Displacements are important for two reasons. Firstly, the serviceability requirement for structures is usually specified as a limit on the magnitude of certain displacements. Secondly, for indeterminate structures, one cannot determine the internal forces using only the equations of equilibrium. 1.5 Basic Analytical Tools of Structural Analysis 41 One needs to consider the displacements and internal forces simultaneously. This topic is addressed in part II of the text. The following example illustrates one of the strategies employed for statically indeterminate structures. Example 1.5: A Statically Indeterminate Beam Given: The beam shown in Fig. E1.5a. Determine: The reactions. Solution: First, we construct the FBD for the beam (Figs. E1.5b and E1.5c). Fig. E1.5b Construction summation of forces in the X and Y directions and summation of moments about A, we obtain the following three equations. X Fx ¼ 0 þ Ax ¼ 0 X Fy "þ Ay þ Cy þ Dy ¼ P X L P ¼ Cy LDy Mabout A ¼ 0) 4 2 We have only two equations for the three vertical reactions, Ay, Cy, and Dy. Therefore, we cannot determine their magnitude using only the force equilibrium equations. The Force (or Flexibility) method for this problem is based on releasing one of the redundant supports, say support C, replacing it with an unknown force, Cy, and allowing point C to move vertically under the action of the applied loads. First, we take Cy ¼ 0, and apply P. Point C moves an amount Δc as shown in the figure below. Then, we take P ¼ 0, Cy ¼ 1, and determine Δc the corresponding v 42 1 Introduction to Structural Engineering movement at C due to a unit upward force at C. Assuming the support at C is unyielding, the net movement must be zero. Therefore, we increase the force Cy until this condition is satisfied. Once Cy is known, we can find the remaining forces using the equations of static equilibrium. In order to carry out this solution procedure, one needs to have a method for computing displacements of beams. These methods are discussed in Chap. 3. 1.6 Summary 1.6.1 Objectives of the Chapter • Provide an overview of the set of issues that a structural engineer needs to address as a practicing professional engineer. • Introduce the basic analytical methods of structural analysis and describe how they are applied to determine the response of a structure. 1.6.2 Key Issues and Concepts Introduced • A structure is an assemblage of structural components which are arranged in such a way that the structure can withstand the action of the loads that are applied to it. Structures are classified according to their makeup such as trusses, frames, and their functions such as bridges, office buildings. 1.6 Summary 43 • The primary concern of a structural engineer is to ensure that the structure will not collapse during its expected lifetime. This requires firstly that the engineer properly identify the extreme loading that the structure is likely to experience over its design life, and secondly, that the structure is dimensioned so that it has adequate capacity to resist the extreme loading. • Structures are restrained against rigid body motions by supports. When the structure is loaded, reaction forces are developed by the supports. A minimum of three nonconcurrent reaction forces are required to prevent initial instability. Initial instability occurs when the reactions are insufficient or they are not properly arranged to prevent rigid body motion. In this case, the structure will undergo an infinitesimal load. This condition can be corrected by modifying the supports or including additional members. • Loss of stability under loading can occur when a primary structural member loses its capacity to carry load due to either elastic buckling or failure of the material. There are two modes of material failure: "brittle" and "ductile." Brittle failure occurs suddenly with a complete loss in load capacity. One should avoid this mechanism. Ductile failure is evidenced by substantial inelastic deformation and loss in stiffness. The limit state design procedure allows for a limited amount of inelastic deformation. • Loads applied to civil structures are categorized according to direction. Vertical loads are due to gravitational forces and are defined in terms of the weight of objects. Lateral loads are produced by natural events such as wind and earthquake. The relative importance of these loads depends on the nature of the structure and the geographical location of the site. • Loads are also generated during the construction of the structure. The design loading for certain types of structures such as segmented concrete girders is controlled by the construction process. Most structural failures occur during the construction process. • Loads are also classified according to the time period over which the loads are applied. Long-term loads, such as self-weight, are called "dead" loads. Loads whose magnitude or location changes are called "live" loads. Typical live loads are produced by vehicles crossing bridges, and people occupying buildings. • Extreme loads such as wind and earthquakes are defined in terms of their return period, which is interpreted as the average time interval between occurrences of the event. One speaks of the 50-year wind, the 50-year earthquake, etc. The magnitude of the load increases with increasing return period. • The design life of a structure is the time period over which the structure is expected to function without any loss in functionality. Bridges are designed to last at least 100 years. Industrial buildings are expected to have design lives usually greater than 100 years. The probability that a structure will experience an extreme event over its design life is approximately equal to the ratio of the design life to the return period. • The effect of wind acting on a building is represented by a pressure loading distributed over the external surfaces. The magnitude and spatial variation of the pressure depends on the shape of the building and the local wind environment. Positive pressure is generated on windward vertical forces and steeply inclined roofs. Turbulence zones occur on flat roof and leeward faces and result in negative pressure. • Design codes specify procedures for computing the spatial distribution of wind pressure given the expected extreme wind velocity at the geographic location. The wind velocity tends to be larger in coastal regions. A typical wind velocity for coastal regions of the USA is 100 miles per hour. The corresponding wind pressure is approximately 20 psf. • Snow loading is represented as a uniform download pressure acting on the roof zones of a structure. Design snow pressure is based on ground snow data for the region where the structure 44 • • • • 1 Introduction to Structural Engineering is located. Snow is an important loading for the northern part of the USA, Canada, and Eastern Europe. Earthquakes produce sudden intense short-term ground motion which causes structures to vibrate. The lateral floor loading is due to the inertia forces associated with the acceleration generated by the ground shaking and is generally expressed as (Wf/g)amax, where Wf is the floor weight, and amax is the peak value of floor acceleration. Seismic engineering is specialized technical area which is beyond the scope of this textbook. However, the reader should be knowledgeable of the general seismic design strategy. Conventional structural design philosophy is based on satisfying two requirements: safety and serviceability. Safety relates to extreme loading and is concerned with preventing collapse and loss of life. Safety is achieved by providing more resistance than is required for the extreme loading. Serviceability relates to loading which occurs during the structure's lifetime. One needs to ensure that the structure remains operational and has no damage. Motion-Based Design, also called performance-based design, is an alternate design methodology that takes as its primary objective the satisfaction of motion-related design requirements such as displacement and acceleration. The goal here is to provide sufficient stiffness and energy dissipation mechanisms to limit the motion under extreme loading. Structural analysis is concerned with quantifying the response of a structure subjected to external loading. This effort involves determining the reactions, internal forces, and displacement profiles. One generally carries out the analysis in two steps: study of forces and study of displacements. In the study of forces, one applies the force equilibrium equations to isolated segments of the structure called FBDs. Selecting appropriate FBDs is a skill acquired through practice. In the study of displacements, one first uses a geometric-based approach to express the deformation measures in terms of displacement measures. The displacement measures are then related to the internal forces by introducing certain material properties such as the elastic modulus. These relations allow one to determine the displacements, given the internal forces. We apply this approach throughout part II of the text. It provides the basis for the analysis of statically indeterminate structures. References 1. Schodek DL, Bechthold M. Structures. Englewood Cliffs: Pearson/Prentice Hall; 2008. 2. Hibbeler RC. 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From a mechanics perspective, they are ideal structures for introducing the concepts of equilibrium and displacement. We deal first with the issues of stability and static determinacy, and then move on to describe manual and computer-based techniques for determining the internal forces generated by external loads. A computational scheme for determining the displacements of truss structures is presented next. Given a structure, one needs information concerning how the internal forces vary as the external live load is repositioned on the structure for the design phase. This type of information is provided by an influence line. We introduce influence lines in the last section of this chapter and illustrate how they are constructed for typical trusses. This chapter focuses on linear elastic structural analysis. Nonlinear structural analysis is playing an increasingly more important rule in structural design. However, we believe an understanding of linear analysis is essential before discussing the topic of nonlinear analysis. 2.1 Introduction: Types of Truss Structures Simple two-dimensional (2-D) truss structures are formed by combining one-dimensional linear members to create a triangular pattern. One starts with a triangular unit, and then adds a pair of members to form an additional triangular unit. This process is repeated until the complete structure is assembled. Figure 2.1 illustrates this process for the case where all the members are contained in a single plane. Such .structures are called plane trusses; the nodes are also called "Joints." Three members connected at their ends form a rigid structure in the sense that, when loaded, the change in shape of the structure is due only to the deformation of the members. It follows that a # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_2 47 48 2 Statically Determinate Truss Structures Fig. 2.1 Simple planar truss construction Fig. 2.2 Simple space truss construction Fig. 2.3 Simple planar trusses structure constructed in the manner described above is also rigid provided that the structure is suitably supported. Simple three-dimensional (3-D) space trusses are composed of tetrahedral units. Starting with a tetrahedral unit, one forms an additional tetrahedral unit by adding three linear elements, as illustrated in Fig. 2.2. When the structure is suitably supported to prevent rigid body motion, the assemblage is rigid. The question of suitable supports is addressed later in the chapter. Examples of simple planar trusses are shown in Fig. 2.3. Starting with the initial triangle abc, one adds the nodes d, e, etc. Trusses may also be constructed by using simple trusses as the "members," connected together by additional members or joints. These structures are called compound trusses. Figure 2.4 shows several examples of compound trusses, where the simple trusses are shown as shaded areas. A truss geometry that does not fall in either the simple or compound category is called a complex truss [1]. Examples are shown in Fig. 2.5. This type of truss is not used as frequently as either simple or compound trusses. 2.1.1 Structural Idealization Trusses are components of larger structural systems, such as buildings, bridges, and towers. In order to evaluate the behavior under loading, one needs to identify the main structural components of 2.1 Introduction: Types of Truss Structures 49 Fig. 2.4 Compound planar trusses Fig. 2.5 Complex planar trusses the system and determine how the external load is transmitted from one component to another. This process is called "structural idealization;" it is a key step in the analysis process. In what follows, we illustrate idealization strategies for typical bridges and roof systems. Figure 2.6 shows a typical single span truss bridge system. The key components are the two simple planar trusses, the lateral bracing systems at the top, sides, and bottom levels and the flooring system consisting of floor stringers/beams and deck slab. Loading applied to the deck slab is transmitted through the stringer/beam system to the bottom nodes of the two planar trusses. The major percentage of the analysis effort is concerned with analyzing a simple truss for dead weight, wind, and traffic loading. Roofing system for buildings such as warehouses, shopping centers, and sports facilities employ trusses to support the roof envelope. Figure 2.7 illustrates a scheme for a typical roofing system for a single-story industrial building. The roof system consists of steel decking attached to purlins which, in turn, are supported at the top nodes of the planar trusses. Loading applied to the roof area in a bay is transmitted through the purlins to the trusses adjacent to the bay, and eventually to the supports. Bracing is incorporated to carry the lateral loading which may act either in the longitudinal or 50 2 Statically Determinate Truss Structures Fig. 2.6 Single span truss bridge system Fig. 2.7 Typical industrial building roofing system transverse direction. The primary effort for this structural system is concerned with analyzing a single planar roof truss for the tributary area loading applied at the top chord nodes. 2.1.2 Historical Background The first application of truss type structures is believed to be in Egyptian boats built between 3100 and 2700 BC. Egyptian boat builders used trusses constructed by tying the members together with vines to form the sides and attached the outer hull to these structures. The Romans used wood trusses for bridges and roofs. Examples are a bridge over the Danube (circa 106 AD) and the entrance to the Pantheon (circa 120 AD). The next time frame is that of the Middle Ages. Examples of trusses are found in English cathedrals (Salisbury Cathedral, circa 1258 AD) and great halls (Westminster Palace, circa 1400). Deployment of wooden trusses continued through the Gothic and Renaissance periods, mainly to support roofs. The Engineers of these time periods intuitively understood the rigidity provided by the triangular form, but lacked a theory that they could apply to evaluate the response for a given load. A major contribution to the theory is the work of Leonardo da Vinci (1452–1519), who formulated the concepts of force and moment as vectors and showed that forces can be combined using a graphical construction that is now called the force parallelogram. From the early 1600s to the 2.1 Introduction: Types of Truss Structures 51 Fig. 2.8 Covered wood bridges mid-1800s, many advances in the development of a scientific basis for a theory of structures were made. Key contributors were Newton (1642–1729), Hooke (1635–1703), Galileo Galilei (first useable formula for strength of a cantilever beam-1638), Euler (theory of buckling of columns1757), Bernoulli (bending deformation of a beam-1741), Navier (produced an integrated theory of Structural Mechanics—1826), and Mobius (published the Textbook of Statics—1837). Wooden bridge truss structures were popular in the early 1800s, especially in the USA. There are many examples of covered wooden bridges in Vermont and New Hampshire. Figure 2.8 illustrates some typical structural schemes. 52 2 Statically Determinate Truss Structures Fig. 2.9 Examples of named trusses There was a flourishing industry in New England producing wooden bridge trusses, many of which were exported to Europe.. As with many emerging technologies, competition from other emerging technologies occurred and eventually took over the market for the product. The first impetus for change was the Industrial Revolution which occurred in the early 1800s. The concept of the railroad was invented during this period. This invention created a demand for more robust and more durable bridges to carry the heavier moving loads over longer spans. Cost also became an issue. Up to this time, wooden bridges were designed to carry light pedestrian and horse and carriage traffic over relatively short spans. Their expected life was short, but since they used local materials and local labor, they were not expensive and durability was not an issue. However, they were not adequate for the railroad traffic and other solutions were needed. Another technology that was evolving in the late 1700s was iron making. Processes for making cast and wrought iron cheaper than existing methods were developed in the 1780s. Methods for shaping wrought iron into shapes that could be used as truss members were also invented simultaneously. These inventions set the stage for the use of iron members in trusses during the early 1800s. Initially, wrought iron was used for tension elements and wood for compression elements. Gradually, cast iron replaced wood for compression elements. The first all iron truss bridge in the USA was built in 1840 by Squire Whipple, a leading bridge designer in the USA at that time. He is also known for his book Essay on Bridge Building, published in 1847, the first publication on Structural Theory by a US author. Some other designers active in the 1840s were W. Howe, T. Pratt, A. Fink in North America, and J. Warren in the UK. Trusses of this era were given the name of the individual who designed or constructed them. Examples are shown in Fig. 2.9. Starting around 1850, iron trusses were used not only for bridges but also for other long-span structures such as market halls, exhibition buildings, and railway stations. Notable examples are the Crystal Palace, the Eiffel tower, and the Saint Pancras station (Fig. 2.10). During the period from 1850 to 1870, an improved version of iron called steel was invented. This material was much stronger than cast iron; more ductile than wrought iron, and quickly displaced iron as the material of choice. The first all steel truss bridge in the USA was built for the Chicago and Alton Railroad in 1879. The structure consisted of a series of Whipple trusses with a total length of 1500 ft spanning over the Mississippi River at Glasgow, Missouri. The first major long-span steel bridge was the Firth of Forth Bridge built in Scotland in 1890. Another similar 2.1 Introduction: Types of Truss Structures Fig. 2.10 Examples of structures made of iron trusses. (a) Crystal Palace. (b) Eiffel Tower. (c) Saint Pancras station 53 54 2 Statically Determinate Truss Structures Fig. 2.11 Typical pin joint connections. (a) Frictionless pin. (b) Bolted connection. (c) Welded connection Fig. 2.12 Example of early steel bridges—Firth of Forth Bridge, Scotland cantilever style truss bridge was built over the St. Lawrence River in Quebec, Canada in 1919. The initial steel structures used eyebars and pins. Rivets replaced pins as connectors in the late 1800s. High-strength bolts and welding are now used to connect the structural members in today's modern steel constructions. Figure 2.11 illustrates typical bolted and welded connections. Connection details are usually designed by the steel fabricator and checked by the structural engineer. The goal in connection design is to minimize steel erection time. Steel truss bridges were the dominant choice for long-span crossings until the mid-1900s when another structural form, the cable-stayed bridge, emerged as a competitor. Cable-stayed bridges have essentially captured the market for spans up to about 900 m. Segmented concrete girder construction has also emerged as a major competitor for somewhat shorter spans. Plane trusses now are used mainly for prefabricated joists, for gable roof systems, and for spanning long interior distances in buildings and sporting facilities such as convention halls and stadiums. Threedimensional space trusses are used in dome type structures such as shown in Fig. 2.13, and also for towers. One of the most notable examples of the space truss concept is the Eiffel Tower in Paris, France. 2.2 Analysis of Planar Trusses 55 Fig. 2.13 Three-dimensional truss roof system 2.2 Analysis of Planar Trusses In this section, we focus on introducing analysis and behavior issues for planar trusses. The discussion is extended in the next section to deal with three-dimensional space structures. The analysis of trusses is based on the following idealizations that ensure that the forces in the members are purely axial: 1. The loads and displacement restraints are applied only at the nodes. 2. The members are connected with frictionless pins so that the members can rotate freely and no moment exists at the ends. 3. The stress due to the weight of the members is small in comparison to the stress due to the applied loads. 4. Each member is straight and is arranged such that its centroidal axis coincides with the line connecting the nodal points. With these restrictions, it follows that a member is subjected only to an axial force at each end. These forces are equal in magnitude and their line of action coincides with the centroidal axis of the member. There is only one unknown per member, the magnitude of the force. The resulting state is uniform axial stress throughout the member. Depending on the loading, the member force may be either tension or compression. Figure 2.14 illustrates free body diagrams for a truss member and its associated nodes. 56 2 Statically Determinate Truss Structures Fig. 2.14 Free body diagram of a truss member and its associated nodes. (a) Tension. (b) Compression Fig. 2.15 Concurrent force system at a node 2.2.1 Equilibrium Considerations The equilibrium requirements .for a body subjected to a non-concurrent force system are specified in Sect. 1.5.2. In general, the resultant force vector and the resultant moment vector with respect to an arbitrary moment center must vanish. One can apply these requirements either to the complete truss or to the individual nodes. Each node of a plane truss is acted upon by a set of coplanar concurrent forces. There are no moments since the pins are frictionless and the lines of action of the forces intersect at the node. For equilibrium of a node, the resultant force vector must vanish. In Squire Whipple's time (1840s), equilibrium was enforced using Leonardo da Vinci's graphical method based on the force polygon. Now, one applies an analytical approach based on resolving the force vectors into components and summing the components. The corresponding scalar equilibrium equations are X X Fn ¼ 0 Fs ¼ 0 ð2:1Þ where n and s are two arbitrary nonparallel directions in the plane. Figure 2.15 illustrates this notation. 2.2 Analysis of Planar Trusses 2.2.2 57 Statically Determinate Planar Trusses In general, three motion restraints are required to prevent rigid body motion of a planar truss. Two of these restraints may be parallel. However, the third restraint cannot be parallel to the other two restraints since, in this case, the truss could translate in the direction normal to the parallel restraint direction. Each restraint generates an unknown force, called a reaction. Therefore, the minimum number of reactions is 3. Examples of typical support motion restraints and the corresponding reactions are shown in Fig. 2.16. There are two scalar equilibrium equations per node for a plane truss. Assuming that there are j nodes, it follows that there are a total of 2j equilibrium equations available to determine the force unknowns. We suppose there are m members and r reactions. Then, since each member and each reaction involves only one unknown force magnitude, the total number of force unknowns is equal to m + r. When the number of force unknowns is equal to the number of equilibrium equations, the structure is said to be statically determinate. If m + r < 2j, the truss is unstable since there are an insufficient number of either member forces or reactions or possibly both to equilibrate the applied loads. It follows that a plane truss is statically determinate., unstable, or indeterminate when m þ r ¼ 2 j Statically determinate m þ r < 2 j Unstable m þ r > 2 j Statically indeterminate ð2:2Þ Fig. 2.16 Types of supports for planar trusses. (a) Hinge support (two restraints) two reactions). (b) Roller Support (one restraint) one reaction). (c) Rigid link 58 2 Statically Determinate Truss Structures A word of caution: a statically determinate truss may also be unstable if the reactions are not properly aligned so as to prevent rigid body motion of the truss. We discuss this point in more detail in the following section. 2.2.3 Stability Criterion In this section, we examine in more detail the question of whether a planar truss structure is initially stable. Assuming a plane truss has m members, r reactions, and j joints, there are 2j force equilibrium equations that relate the known (given) joint loads and the (m + r) unknown forces. If m + r < 2j, the number of force unknowns is less than the number of equilibrium equations that the forces must satisfy. Mathematically, the problem is said to be underdetermined or inconsistent. One cannot find the exact solution for an arbitrary loading, except in the trivial case where the magnitude of all the loads is zero, and consequently the forces are zero. Once a nontrivial load is applied, the structure cannot resist it, and motion ensues. The descriptor "initial instability" is used to denote this condition. Even when m + r ¼ 2j, a truss may still be unstable if the motion. restraints are not properly arranged to prevent rigid body motion of the structure. There may be an insufficient number of restraints or the restraints may be aligned in such a way that rotation of a segment is possible. The stability of a complex truss depends on the geometrical arrangements of the members. Even though the truss satisfies the condition, m + r ¼ 2j, and has sufficient restraints, it still may be unstable. The instability condition becomes evident when one attempts to determine the member forces using the 2j force equilibrium equations. The solution is not unique when the structure is unstable. When m + r > 2j and the structure is suitably. restrained against rigid body motion, the structure is said to be statically indeterminate. This terminology follows from the fact that now there are more force unknowns than equilibrium equations, and not all the forces can be determined with only equilibrium considerations. One needs some additional equations. We address this type of problem in Part II. In what follows, we illustrate the initial stability criteria with typical examples. Stability can be defined in a more rigorous way using certain concepts of linear algebra, a branch of mathematics that deals with linear algebraic equations. This approach is discussed in Sect. 2.6 Example 2.1 Simple Trusses Given: The structures. shown in Fig. E2.1a–d Fig. E2.1 Determine: Whether the structures are initially stable, determinate, or indeterminate. Solutions: Case (a): There are five members, three reactions, and four nodes. Then applying (2.2) 2.2 Analysis of Planar Trusses 59 mþr¼8 2j ¼ 8 The structure is determinate and initially stable. Case (b): mþr¼9 2j ¼ 8 There is one extra force and therefore the structure is initially stable and indeterminate to the first degree. Case(c): The stability criterion appears to be satisfied. mþr¼8 2j ¼ 8 However, the number of supports is insufficient to prevent rigid body motion in the horizontal direction. Therefore, the structure is initially unstable. 60 2 Statically Determinate Truss Structures Case (d): The stability criterion appears to be satisfied. mþr¼8 2j ¼ 8 However, the three displacement restraints are concurrent (point A), and therefore the structure can rotate at point A. It follows that the structure is initially unstable. Example 2.2 A Compound Truss Given: The structure shown in Fig. E2.2a. This compound truss is actually a combination of two simple trusses. Fig. E2.2a Determine: Is the structure statically determinate? Solution: The structure is statically determinate and stable. m þ r ¼ 24 2j ¼ 24 Example 2.3 A Complex Truss 2.2 Analysis of Planar Trusses 61 Given: The complex truss defined in Fig. E2.3a. Fig. E2.3a Determine: Is the truss statically determinate? Solution: There are three restraints, six joints, and nine members. m þ r ¼ 12 2j ¼ 12 The truss appears to be stable. Note that the condition, m + r ¼ 2j is necessary but not sufficient to ensure stability of this truss. Sufficient conditions are discussed further in Sect. 2.6. In what follows, we describe two classical hand computation-based procedures for finding the member forces in simple and compound trusses due to an applied loading. .These approaches are useful for gaining insight as to how loads are carried by structures. That is the most important aspect of structural engineering that one needs to master in order to be a successful practitioner. Also, although most current structural analysis is computer based, one still needs to be able to assess the computer-generated results with a simple independent hand computation. 2.2.4 Method of Joints: Planar Trusses Each joint of a plane truss is subjected to a concurrent force system. Since there are two equilibrium equations for a 2-D concurrent force system, one can solve for at most two force unknowns at a particular joint. The strategy for the method of joints is to proceed from joint to joint, starting with the free body diagram of a joint that has only two unknowns, solving for these unknowns, and then using this newly acquired force information to identify another eligible joint. One continues until equilibrium has been enforced at all the joints. When all the joints are in equilibrium, the total structure will be in equilibrium. This analysis procedure was first described in Squire Whipple's 1840 Essay on Bridge Building. Enforcing the equilibrium conditions is simplified if one works with the force components referred to a common reference frame. Once one component is known, it is a simple step to determine the magnitude of the other component and the force. As an illustration, we consider the member shown in Fig. 2.17. The ratio of force components is equal to the ratio of the projected lengths. This equality follows from the fact that the direction of the force is the same as the direction of the line. Here, we are taking the horizontal and vertical directions as the common reference frame. 62 2 Statically Determinate Truss Structures Fig. 2.17 Force components Fig. 2.18 Zero force member Fy Ly ¼ ¼ tan θ Fx Lx Similarly, the force is determined using F¼ Fy Fx ¼ cos θ sin θ Another simplification is possible when the joint has only three members, two of which are colinear, and there is no applied load at the joint. Figure 2.18 illustrates this case. There is only one force acting at an angle to the direction of the two common members, and equilibrium in the normal direction (n) requires the magnitude of this force to be zero. The other two forces must have the same magnitude. When applying the method of joints, it is convenient to first determine the reactions by enforcing global equilibrium on the total structure. With the reactions known, it may be easier to locate a joint having only two unknown member forces. In what follows, we present a set of examples that illustrate how the method of joints is efficiently applied. Example 2.4 Three-Member Truss Analyzed by the Methods of Joints Given: The truss and loading defined by Fig. E2.4a. 2.2 Analysis of Planar Trusses 63 Determine: The reactions and member forces. Fig. E2.4a Solution: We first find the reactions at joints a and b. Moment summation about joint a leads to the y reaction at b. Force summations provide the remaining two reaction forces. The results are shown in Fig. E2.4b. X X FX ¼ 0 ! þ Rax þ 10 ¼ 0 FY ¼ 0 " þ Ray þ 15 2:5 ¼ 0) Rax ¼ 10) ∴ Rax ¼ 10 kip Ray ¼ 12:5 ∴ Ray ¼ 12:5 kip # Fig. E2.4b Reactions One can start at any joint since they all have just two unknown member forces. We pick joint b (Fig. E2.4c). Summation of forces in the y direction gives Fbc,y ¼ 2.5 kip. Then, summing forces in the x direction requires Fba being compressive and equal to 2.5 kip. We indicate a tensile member force with an arrow pointing away from the joint. The opposite sense is used for compression. One converts the force components to the force magnitude using the Pythagorean Theorem, qffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi F ¼ F2x þ F2y . X Fy ¼ 0 Fbc, y ¼ 2:5 " Then Fbc, x ¼ 2:5 Fx ¼ 0 Fba ¼ 2:5 kip ðcompressionÞ pffiffiffi ∴ Fbc ¼ 2:5 2 kip ðtensionÞ 64 2 Statically Determinate Truss Structures Fig. E2.4c Joint b There is only one unknown member force left, Fca. One can work with either joint a or joint c. We pick joint c. The free body diagram for joint c is shown in Fig. E2.4d. Equilibrium in the x direction requires Fca,x ¼ 12.5 kip. X pffiffiffi Fx ¼ 0 Fca, x ¼ 12:5 ∴Fca ¼ 12:5 2 kip ðtensionÞ Fig. E2.4d Joint C Note that in this example we do not need to use the remaining equilibrium equations (one for joint c and two for joint a) since we used instead three global equilibrium equations to calculate the reactions. The total number of joint equilibrium equations is equal to six (two per joint three joints). If we use three equations for global equilibrium, there are only three independent equations left to apply to the joints. The final results are shown on the sketch below. Tensile forces are denoted with a + sign, and compressive forces with a sign. 2.2 Analysis of Planar Trusses 65 Fig. E2.4e Example 2.5 Five-Member Truss Analyzed by the Methods of Joints Given: The truss defined in Fig. E2.5a. Determine: The reactions and member forces for the loading shown. Fig. E2.5a Solution: We first find the reactions and then proceed starting with joint a, and then moving to joints c and d. 66 2 Statically Determinate Truss Structures Fig. E2.5b SMa = 0 Fig. E2.5c + – 27(4) – 18(8) – Rc(8) = 0 Þ Rc = – 31.5 \ Rc = 31.5 kip ˉ 2.2 Analysis of Planar Trusses 67 Fig. E2.5d Joint a 8P < Fx ¼ 0 Fad cos α þ Fab cos β þ 18 ¼ 0 :P Fig. E2.5e Joint c Fy ¼ 0 Fad sin α þ Fab sin β þ 4:5 ¼ 0) Fad ¼ 31:5 kN Fab ¼ 16:1 kN 68 2 8P < Fx ¼ 0 :P Fy ¼ 0 Fcd cos α þ Fcb cos β ¼ 0 Fcd sin α þ Fcb sin β 31:5 ¼ 0 Statically Determinate Truss Structures) Fcd ¼ 31:5 kN Fcb ¼ 56:35 kN Fig. E2.5f Joint d X Fy ¼ 0 Fbd ¼ 37:8 kN ðcompressionÞ The final forces are listed below. Example 2.6 Five-Member Truss Analyzed by the Methods of Joints Given: The truss defined in Fig. E2.6a. Determine: The reactions and member forces for the loading shown. 2.2 Analysis of Planar Trusses 69 Fig. E2.6a Bridge truss Solution: We note that the structure and loading are symmetrical with respect to a vertical axis through points e and d. It follows that the forces in symmetrically located members are equal, and therefore we need to find the forces in only ½ of the structure. Joints c, e, and f are special in the sense that two incident members are colinear. Then, noting Fig. 2.18, Fcb ¼ 10 kN ðtensionÞ Fed ¼ 0 Ffg ¼ 10 kN ðtensionÞ Fig. E2.6b There are multiple options. We can first find the reactions and then proceed inward, starting with joint a, and then moving to joints c and b. An alternate approach would be to start at joint d, find the y component of Fbd, and then move to joint b. Fig. E2.6c Reactions We list the results for the first approach below. We first find Fba,y with the vertical equilibrium condition at joint a. Then, we find Fac from the horizontal component of Fba. 70 2 X X

Fy ¼ 0 Fx ¼ 0 Statically Determinate Truss Structures Fba, y ¼ 15 kN # Then Fba, x ¼ 15 kN ! pffiffiffi ∴ Fba ¼ 15 2 kN ðcompressionÞ Fac ¼ 15 kN ðtensionÞ Fig. E2.6d Joint a At joint c, we note from the sketch that Fdc ¼ 15 kN (tension). Fig. E2.6e Joint c At joint b, we note from the sketch that Fdb must be in tension and Fbe must be in compression. Fig. E2.6f Joint b We first find Fdb, y with the vertical equilibrium condition at joint b. Then, we apply the horizontal equilibrium equation at joint b. X Fx ¼ 0 Fdb, y ¼ 5 # pffiffiffi Then, Fdb, x ¼ 5 ∴ Fdb ¼ 5 2 kN ðtensionÞ Fbe ¼ 20 kN ðcompressionÞ 2.2 Analysis of Planar Trusses 71 The resultant member forces are shown below. Note that, for this loading, the members in the top zone (the top chord) are in compression and the bottom chord members are in tension. The interior vertical and diagonal members are all in tension. When iron was used as a structural material, cast iron, which is relatively weak in tension, was employed for the top chord members and wrought iron, which is relatively strong in tension, for the verticals, diagonals, and bottom chord members. Fig. E2.6g If the truss structure is inverted as shown below, the sense of the member forces is also reversed. This geometric arrangement is preferred for bridge crossings when the clearance below the structure is not a problem. Fig. E2.6h Example 2.7 A Cantilever Truss Analyzed by Method of Joints Given: The truss and loading defined by Fig. E2.7a. Determine: The member forces for the loading shown. 72 2 Statically Determinate Truss Structures Fig. E2.7a Solution: We start at joint c. Starting at joint c, we observe that Fcb ¼ 0. Then, moving to joint b, it follows that Fbe ¼ 0. Fig. E2.7b Zero force members In this case, we do not need to first find the reactions. We can start at joint a. Fig. E2.7c Joint a X Fy ¼ 0 Fba, y ¼ 40 ! Fba, x ¼ 40 · Fba ¼ 50 kip ðtensionÞ 2 Analysis of Planar Trusses Fac ¼ 30 kip ðcompressionÞ Next, we determine Fdb and Fgf. Fig. E2.7d Joint d X Fx ¼ 0 Fy ¼ 0 Fde x ¼ 40 With Fdf known, we can determine Fde X Fx ¼ 0 At joint e, we determine Fef and Feg. Fig. E2.7e Joint e ∴Fdf , y ¼ 40 Fdf ¼ 40 5 kip ðtensionÞ Fde ¼ 110 kip ðcompressionÞ 74 2 Statically Determinate Truss Structures The last joint is joint f. We first determine Ffg,x X Fx ¼ 0 Fy ¼ 15 ∴Ffg ¼ 25 kip ðcompressionÞ Then, X Fx ¼ 0 Ffh ¼ 100 kip ðtensionÞ Fig. E2.7f Joint f The final forces are listed below. Fig. E2.7g 2.2 Analysis of Planar Trusses 75 Example 2.8 Gable Roof Truss Analyzed by the Method of Joints Given: The truss and loading defined by Fig. E2.8a. Determine: The member forces. Fig. E2.8a Solution: Fig. E2.8a shows a typical truss structure for supporting roof (top joints) and ceiling (bottom joints) loads. Members cb and gf function to transfer loads to the top joints (b and f). Their force magnitudes are Fbc ¼ 5 kN ðtensionÞ Fgf ¼ 5 kN ðtensionÞ All the remaining joints have at least three unknown member forces and reactions. Therefore, we start the analysis by first finding the reactions. Fig. E2.8b Reactions Given the reactions, we start at joint a. Force Fba must be compressed and Fba,y ¼ 22.5 #. Then, Fba,x ¼ 22.5 and Fba ¼ 31.8 kN. It follows that Fac is in tension and equal to 22.5 kN. 76 2 Statically Determinate Truss Structures Fig. E2.8c Joint a We then move on to joint b. Members ab and bd are colinear, and member be is normal to this common direction. Summing forces in the normal direction results in pffiffiffi X Fn ¼ 0 Fbe ¼ 0 þ 5ð Þ cos α ¼ 15 kN ðcompressionÞ 2 Next, summing forces in the tangential direction leads to Fbd. X pffiffiffi Ft ¼ 0 Fbd ¼ 22:5 2 δ10 þ 5Þ cos α ¼ 15 2 kN ðcompressionÞ Fig. E2.8d Joint b The last force is Fde. We use joint d shown in Fig. E2.8e. Summing forces in the y direction leads to Fde ¼ 20 kN (tension) 2 Analysis of Planar Trusses 77 Fig. E2.8e Joint d The final forces are listed below. Fig. E2.8f 2.2.5 Method of Sections If one wants to determine the force in a particular member, applying the method of joints might not be convenient since in general it involves first finding the force in other members. For example, consider the truss shown in Fig. 2.19a. Suppose the force in member ef is desired. One possible strategy is to first determine the reactions at joint a, then proceed to joints b, c, d, and lastly e, where the Y component of Fef can be determined once Fed is known. Another possible strategy is to start at joint j, and then proceed to joints i, h, g, and f. Either approach requires some preliminary computation that provides information on forces that may or may not be of interest. The method of sections is an analysis procedure that avoids this preliminary computation. One passes a cutting plane through the structure, isolates either the left or right segment, and applies the equilibrium equations for a rigid body to the segment. The choice of cutting plane is critical. It must cut the particular member whose force is desired, and other members that are concurrent. This restriction is necessary since there are only three equilibrium equations for planar loading, and therefore, one can only determine three unknowns. We illustrate this method for the truss defined in Fig. 2.19a. To determine Fef, we use the vertical cutting plane 1-1 and consider the left segment shown in Fig. 2.19c. Summing forces in the Y direction leads to: 78 2 Statically Determinate Truss Structures Fig. 2.19 (a) An example of a truss. (b) Cutting vertical plane. (c) Truss segment for method of sections X Fy ¼ 0 " þ Fef cos α ¼ P1 P 2 Ray δ2:3Þ We point out here that the function of the diagonal members is to equilibrate the unbalanced vertical forces at the sections along the longitudinal axis. These forces are called "shear" forces. If the force in member ed is desired, one can use the moment equilibrium condition with respect to point e which is the point of concurrency for Fef and Ffd. Summing moments about e leads to X Mabout e ¼ 0 hFdf ¼ 2hRay 2P1 ϐ2 δ2:5Þ For parallel chord structures (top and bottom chords are parallel), the function of the chords is to equilibrate the unbalanced moments at the various sections. One chord force is compressive, the other force is tensile. For downward vertical loading, the top chord is generally in compression, and the bottom chord is in tension. The method of sections is convenient in the sense that it allows one to easily identify the sense of a particular member force. 2.2 Analysis of Planar Trusses 79 Example 2.9 Application of the Method of Sections to a Parallel Chord Truss Given: The structure and loading shown in Fig. E2.9a Determine: The force in members Fcd, Fgf, and Fdc. Fig. E2.9a Solution: We start by determining the reactions. SMa = 0 = 2(3) + 4(6) + 3(9) − Re(12) = 0 Re = 4.75 kN Fig. E2.9b Then, we pass a vertical cutting plane through the panel between joints d and c and consider the left segment. Enforcing equilibrium leads to: Fig. E2.9c 80 2 X Fy ¼ 0 Fgf ¼ 2:875 and Fgf ¼ 2:875 ¼ 4:25 kN (tension) X Mat g ¼ 0 4:25ð4Þ 2ð3Þ þ 4:25ð6Þ ¼ 0 Fcd ¼ 9:125 Therefore, Fcd ¼ 9.125 kN (compression) X Fx ¼ 0 Fgf ¼ 9:125 þ 2:875 ¼ 12 ... Therefore, Fgf ¼ kN þ5:375 kN (tension). Fig. E2.9d Example 2.10 The Method of Sections Applied to a Roof Truss Given: The structure and loading shown in Fig. E2.10a. Determine: The member forces Fdb, Fbe, and Fce. Fig. E2.10a Solution: Fig. E2.10b Fig. E2.10b 2.2 Analysis of Planar Trusses 81 Fig. E2.10b To determine the member forces Fdb, Fbe, and Fce, we use a vertical cutting plane. The appropriate segment is shown in Fig. E2.10c. Various options are possible. We choose first to determine Fbe by summing moments about e. Then, summing moments about b, we can find Fce by using either X or Y forces. Fig. E2.10c The calculations for this analysis procedure are listed below: 2 SMb = 0 P FY ¼ 0 " þ Fbe, y ¼ 6 þ Statically Determinate Truss Structures − 2Fce + (18.5) 4 = 0 Fce = 37 kN (Tension) Fbe, y 12 12.5 þ 18:5 ¼ 0 Fbe ¼ 13:41 kN ðcompressionÞ Fbe ¼ 4 sin α Example 2.11 Analysis of K-Type Trusses with the Method of Sections Given: The truss defined in Fig. E2.11a. Determine: The member forces Fab, Fbe, Fed, and Fcd. Fig. E2.11a Solution: We first determine the reactions first. A vertical section such as ①—① cuts four unknown forces and does not lead to a solution. There are no cutting planes that involve only three unknown forces. Therefore, one has to be more creative with the choice of planes. For this type of truss, plane ②—② is the appropriate choice. Isolating the left segment and summing moments about joint e results in Fab. Fig. E2.11b 2 Analysis of Planar Trusses 83 Fig. E2.11c Then summing X forces, X Fx ¼ 0 Fcd ¼ Fab ¼ 6 ∴Fcd ¼ 6 kN ðtensionÞ The force in members Feb and Fed are found using section ①—①. Summing moments about joint d leads to Feb: SMd = 0 = 4(Fab + Feb,x) −10(8) + 8(12) = 0 4(−6 + Feb,x) = 16 = 0 Feb, x ¼ 4 þ2 ∴Feb ¼ 2:24 kN ðtensionÞ cos α Fig. E2.11d We find Fed by summing x forces, and noting that the horizontal components of the diagonal forces must cancel. X Fx ¼ 0 Fed, x ∴Fed ¼ 2:24 kN ðcompressionÞ Fig. E2.11e 84 2 Statically Determinate Truss Structures If one wants all the member forces, one can apply multiple cutting planes or combinations of the method of joints and method of sections. How one proceeds is a matter of personal preference. Example 2.12: A Hybrid Analysis Strategy Given: The truss defined in Fig. E2.12a Determine: All the member forces using a combination of the method of joints and the method of sections. Fig. E2.12a Solution: We note that the structure and loading are symmetrical with respect to a vertical axis through points c and g. It follows that the forces in the symmetrically located members are equal, and therefore we need to find the forces in only ½ of the structure. We start by determining the reactions. The force in members Fbc, Fhc, and Fhg can be determined by passing vertical cutting plane 1-1 and enforcing the equilibrium equations. Fig. E2.12b 2.2 Analysis of Planar Trusses 85 Considering the left segment and enforcing equilibrium leads to: Section 1-1: X pffiffiffiffiffiffi Mat h ¼ 0 ðFbc cos ð45 þ ð6Þ 5 ¼ 1588Þ ¼ 0 Fbc ¼ 10 5 kip ðcompressionÞ X pffiffiffi Fy ¼ 0 Fhc ¼ 14 2 kip ðtensionÞ X Fx ¼ 0 Fhg ¼ 16 kip ðtensionÞ We then enforce equilibrium at joints a and h. Equilibrium at joint a: Fig. E2.12c Joint a ... Fab ¼ 25 kip ðcompressionÞ Fx ¼ 0 Fah ¼ 20 kip ðtensionÞ Equilibrium at joint h: X Fy ¼ 0 Fbh ¼ Fch, y ¼ 4 kip ðcompressionÞ The final member forces are listed below. 86 2 Statically Determinate Truss Structures Fig. E2.12c 2.2.6 Complex Trusses Complex trusses are defined as truss structures that cannot be classified as either simple or compound trusses. In order to determine the member forces, one has to establish the complete set of nodal force equilibrium equations expressed in terms of the member forces. If the truss is statically determinate, the number of equilibrium equations will be equal to the number of force unknowns, and theoretically one can solve these equations for the force unknowns. However, if one cannot solve the equations, the structure is geometrically unstable. In what follows, we expand on this point. Consider the planar truss shown in Fig. 2.20a. There are nine members, three reactions, and six nodes. Then, 2j ¼ 12 m þ r ¼ 9 þ 3 ¼ 12 and the truss is statically determinate. It also has a sufficient number of reactions to prevent rigid body motions. We use 3 of the 12 equilibrium equations to determine the reactions, leaving 9 equations available to solve for the 9 member forces. ... 2.2 Analysis of Planar Trusses 87 Fig. 2.20 (a) Planar truss geometry. (b) Reactions. (c) Joint equilibrium. (d) Modified geometry. (e) Joint equilibrium. (f) Member forces Enforcing equilibrium at joints 2-6 results in the following nine equations: ... The coefficient matrix, B is singular (the determinant of B equals 0). Therefore, a unique solution for the unknown forces does not exist for an arbitrary nodal load. The truss is said to be geometrically unstable since the elements of B depend only on the geometric pattern. In order to eliminate the instability, one needs to modify the geometry. We modify the truss by changing the vertical position of node 3 as shown in Fig. 2.20d. The individual nodal force systems are defined in Fig. 2.20e and the corresponding equilibrium equations are listed in (2.8). Note the change in the coefficients. 2.2 Analysis of Planar Trusses ... In this case, the coefficient matrix B is nonsingular (det B 6¼ 0), and it follows that the truss is geometrically stable ... Solving (2.8) using a computer software system [2] leads to the member forces listed below. ... Then, for example, if P ¼ 10 kN and h ¼ 6 m, the member forces are listed in Fig. 2.20f. Assembling the nodal force equilibrium equations is a tedious operation, especially for three-dimensional space structures. The process can be automated by using matrix operations. We will describe one approach later in Sect. 2.6. 90 2 Statically Determinate Truss Structures 2.3 Computation of Deflections 2.3.1 Introduction Statically Determinate Truss Structures The deflection of the joints are due to the change in length of the members that make up the truss. Each member is subjected to an axial force which produces, depending on the sense, either an extension or a contraction along the longitudinal axis. We call these movements "axial deformation." The study of deflection involves two steps. Firstly, we determine the axial deformation due to the applied loading. This step involves introducing the material properties for the members. Secondly, we need to relate the deflections to the axial deformations. This step is purely geometric. In what follows, we develop procedures for determining the axial deformation due to an axial force, and the joint deflections due to a set of axial deformations. The latter procedure is carried out first using a manual computational scheme. A computer-based scheme is described in the last section of this chapter. 2.3.2 Force-Deformation Relationship Consider the axially loaded member shown in Fig. 2.21. We suppose an axial force, F, is applied, and the member extends an amount e. Assuming the material is linear elastic, e is a linear function of F. We estimate the proportionality factor by first determining the stress, then the strain, and lastly the extension. We discussed this approach in Chap. 1. Stress σ ¼ F A where A is the cross-sectional area 2. Strain ε¼ σ E ¼ F AE where E is young's modulus 3. Extension e ¼ Lε ¼ where L is the member length Fig. 2.21 Axially loaded member FL AE 2.3 Computation of Deflections 91 The member may also experience an extension due to a temperature change or a fabrication error. Introducing these additional terms, the total extension is expressed as e ¼ eforce þ etemperature þ efabrication error δ2:9Þ where eforce ¼ FL AE etemperature ¼ α ΔTL eforce ¼ efabrication error ¼ e0 α is the coefficient of thermal expansion, ΔT is the temperature change, and e0 represents the fabrication error. The total extension, e, is the quantity that produces the displacement of the node. 2.3.3 Deformation–Displacement Relations Consider the planar truss shown in Fig. 2.22. Suppose the members experience deformation and one wants to determine the final position of node B. Our approach is based on first temporarily disconnecting the members at B, allowing the member deformations to occur, and then rotating the members such that they are reconnected. The movements of the nodes from the original configuration to the new configuration are defined as the displacements. These quantities are usually referred to as u and v for a global reference frame having axes X and Y and corresponding displacement components u and v. For structural materials such as steel, the extensions are small in comparison to the original length. Then the rotations will also be small. Noting Fig. 2.22b, and when one assumes, it follows that the displacements are related to the deformations by u ¼ eAB v ¼ eBC δ2:10Þ The simplicity of this results is due to the fact that the structure's geometry is simple (the members are orthogonal to the coordinate axes). We consider the single member AB defined in Fig. 2.23. Our strategy is to track the motion of the end B as it experiences an extension, e. The final length is (L + e) where e is the extension. We assume Δθ is small and project the final length onto the original direction. This step provides a first-order estimate in terms of the nodal displacements. Fig. 2.22 Initial and deformed geometries. (a) Initial geometry. (b) Deformed configuration 92 2 Statically Determinate Truss Structures Fig. 2.23 Extension displacement relationship. Fig. 2.24 Geometry—two member planar truss e ¼ u cos θ þ v sin θ δ2:11Þ We consider next a two-member planar truss shown in Fig. 2.24. Since the member orientations are arbitrary, the deformation–displacement relations will involve all the displacement components. Applying (2.11) to the above structure leads to e1 ¼ u cos θ1 þ v sin θ1 e2 ¼ u cos θ2 þ v sin θ2 δ2:12Þ Given the member extensions, one computes the extensions e1 and e2 and finally determines the displacements by solving (2.12). ... 2.3 Computation of Deflections 2.3.4 93 Method of Virtual Forces The formulation described in the previous section is convenient for manual computation, even for fairly simple trusses. However, there is an alternative procedure called the Virtual Force Method. Engineers prefer this approach since it is based on executing a set of force equilibrium analyses, a task that they are more familiar with. The Method of Virtual Forces is a procedure for determining the deflection at a particular point in a structure given that the member forces are known. A general proof of the method can be found in [3]. We apply the method here for truss type structures. Later in the following chapters, we apply the procedure to beam and frame type structures. The method is restricted to static loading and geometrically linear behavior, i.e., where the displacements are small. This is not a serious restriction for civil structures such as building and bridges. Consider a typical truss shown in Fig. 2.25a. Suppose the deflection, dA, at a specified direction at point A is desired. One applies a virtual force, δPA, at A in the specified direction and computes the corresponding member forces, δF, and reactions, δR, using only the static equilibrium equations. Usually, one takes δPA to be a unit load. Note that this virtual force system is "specialized" for the particular displacement that one is seeking. The displacement is given by the following expression: X X dA δPA ¼ eδF þ d δR δ2:13Þ members where e is the total extension defined by (2.9), d is the support movement, and δR the corresponding reaction. When the supports are unyielding, d ¼ 0, and the statement simplifies to Fig. 2.25 (a) Desired deflection—actual force system F. (b) Virtual force system δF 94 2 X d δPA ¼ eδF δ2:14Þ members Given the actual forces, one evaluates e with (2.9), then determines the product, e δF, and lastly sums over the members. Applying (2.13) is equivalent to solving the set of simultaneous equations relating the displacements and extensions. The following example illustrates this point. Example 2.13 Computation of Deflection—Virtual Force Method Given: The plane truss shown in Fig. E2.13a. Assume A ¼ 1300 mm2 and E ¼ 200 GPa for all members. Determine: The horizontal displacement at node c (uc) Fig. E2.13a Geometry and loading Solution: Applying (2.14), the horizontal displacement at node c (uc) is determined from X X FL uc δP ¼ eδF ¼ AE δF The actual and virtual forces are listed below. Fig. E2.13b Actual forces, F 2.3 Computation of Deflections 95 Fig. E2.13c Virtual forces, δF Using this data, and assuming AE is constant, the computation proceeds as follows: Member ab bc ac L (m) 10 5 3 F (kN) 50 40 80 δF 1 ... The plus sign indicates the deflection is in the direction of the unit load. For A ¼ 1300 mm2, E ¼ 200 GPa, and l ¼ 3 m, the displacement is ... We point out that the virtual force system (δF) results identify which member deformations contribute to the displacement. In this case, only two members contribute to the horizontal displacement. Example 2.14 Computation of Deflection—Virtual Force Method Given: The plane truss shown in Fig. E2.14a. Assume E ¼ 200 GPa. 96 2 Statically Determinate Truss Structures Determine: The value of A required to limit the horizontal displacement at node e to 20 mm. Assume AE is constant for all members. Fig. E2.14a Geometry and loading Solution: Using (2.14) the horizontal displacement at node e (ue) is determined with X X FL ue δP ¼ eδF ¼ AE δF The actual and virtual forces are listed below. Fig. E2.14b Actual forces, F Fig. E2.14c Virtual forces, δF 2.3 Computation of Deflections 97 Using this data, and assuming AE is constant, the computations are carried out below: ... The plus sign indicates the deflection is in the direction of the unit load. For E ¼ 200 GPa, and l ¼ 3000 mm, the required area is ... Example 2.15 Computation of Deflection—Virtual Force Method Given: The plane truss shown in Fig. E2.15a. Assume A ¼ 3000 mm2 and E ¼ 200 GPa for all members. Determine: The vertical displacement at c (vc) due to the loading shown and a settlement of 10 mm at support a. Fig. E2.15a Geometry and loading Solution: Using (2.13), ... 98 2 Statically Determinate Truss Structures ... Example 2.16 Computation of Deflection—Virtual Force Method Given: The plane truss shown in Fig. E2.16a. Member bc and cf also have a fabrication error of +0.5 in. Determine: The vertical component of the displacement at joint g (vg). Take A ¼ 2 in.2 and E ¼ 29,000 ksi for all members. Fig. E2.16a Solution: Fig. E2.16b Actual forces, F Fig. E2.16c Virtual forces, δF Using this data, the computations are carried out below: ... Example 2.17 Deflection of a Gable Truss Given: The plane truss shown in Fig. E2.17a. The truss has variable cross sections, such that A ¼ 6500 mm2 for top chord members, A ¼ 3900 mm2 for bottom chord members, A ¼ 1300 mm2 for diagonal members, A ¼ 650 mm2 for vertical members, l ¼ 3 m, P ¼ 10 kN, and E ¼ 200 GPa. 2.3 Computation of Deflections 101 Determine: The vertical displacement of node j (vj) and the horizontal displacement of node g (ug). Fig. E2.17a Geometry and loading Solution: Fig. E2.17b Actual forces F Fig. E2.17c Virtual forces δFu Fig. E2.17d Virtual forces δFu The computations are organized using the spreadsheet format listed below. Note that the upper and lower chords and only the central member contribute to the central vertical deflection. And only the lower chord contributes to the horizontal support deflection. The plus sign indicates the deflection is in the direction of the unit load. ... We point out that the distribution of member forces depends on the orientation of the diagonal members. Suppose we rotate the diagonals as shown in Fig. E2.17e. The corresponding virtual force system is shown in Fig. E2.17f. Fig. E2.17e Diagonal pattern reversed—actual forces F Fig. E2.17f Diagonal pattern reversed—virtual forces δFv ... Using the data tabulated above, the mid-span deflection calculations are ... The examples presented to this point have been concerned with loads. Structures are also subjected to seasonal (and daily) temperature changes and it is of interest to determine the corresponding nodal displacements. A truss structure responds to a temperature change by adjusting its geometry in such a way that there are no forces introduced in the members. From a design perspective, this behavior is very desirable since member forces, i.e., stresses, are due only to the loads. However, one may need to compute the deflected shape due to temperature change from some initial state. The effect of temperature change is to produce an additional extension in a truss member given by: etemperature ¼ α ΔT L where α is a material property, defined as the coefficient of thermal expansion, and ΔT is the temperature change from the initial state. Then, the form of the Principle of Virtual Force specialized for only temperature and unyielding supports reduces to X d δP ¼ etemperature δF δ2:15Þ The computational procedure is similar to the approach discussed earlier. We evaluate α ΔT L for each member. Then, given a desired deflection, we apply the appropriate virtual loading and compute δF for the members. Lastly, we evaluate the summation. The following example illustrates the details. This discussion applies only for statically determinate structures. Introducing a temperature change introduces internal forces in statically indeterminate trusses. Analysis procedures for this case are discussed in Chaps. 9 and 10. 104 2 Statically Determinate Truss Structures Example 2.18 Computation of Deflection Due to Temperature Given: The plane truss shown in Fig. E2.18a. Determine: The vertical displacement at joint d due to temperature increase of ΔT ¼ 65 F in all the members. Assume A ¼ 2 in.2, E ¼ 29,000 ksi, and α ¼ 6.5 (106)/ F. Fig. E2.18a Solution: The corresponding virtual force system is shown in Fig. E2.18b Fig. E2.18b Virtual forces δF ... 2.4 Influence Lines Consider the plane bridge truss shown in Fig. 2.26a. To design a particular member, one needs to know the maximum force in the member due to the design loading. The dead loading generally acts over the entire span. However, the live loading, such as a truck, can act anywhere on the structure and therefore one needs to determine the location of the live loading that produces the maximum force. A systematic approach to locating the critical position of the live loading is based on first constructing an influence line for the member force. This construction involves a series of analyses, one for each possible location of the live load. The live load is usually taken as a single force, of unit magnitude, which is moved from node to node across the structure. The resulting influence line is a plot of the member force as a function of the location of the applied load. Figure 2.26b illustrates the possible nodal positions of a vertical load applied to the bottom chord, and the corresponding influence line for the force in member "bc". Given this data, one can construct an influence line for any of the member forces. The process described above assumes the loading is a concentrated load applied at the nodes. For bridge structures, the live loading is actually applied to the deck which transmits the load to the transverse beams, and finally to the nodes. The deck is usually simply supported on the transverse beams, so the complete deck-beam system is statically determinate and one can determine the reactions at the nodes using only the equations of statics. We illustrate this computation using the structure shown in Fig. 2.27a. We suppose a truck loading is passing over the span. Consider the position shown in Fig. 2.27b. The wheel loads act on the deck segments fe, and fe. The live load vehicle analysis reduces to just applying loads to the nodes adjacent to the vehicle since the deck segments (gf and fe) are simply supported. Noting Fig. 2.27c, the equivalent nodal forces are ... Note that the reactions are linear functions of x, the position coordinate for the truck. We define Fj as the force in member j. Applying separate unit loads at nodes g and f leads to Fjg and Fjf. Then, according to the equations listed above, the force due to a unit load at x is ... 106 2 Statically Determinate Truss Structures Fig. 2.26 (a) Truss geometry. (b) Load. positions and corresponding member forces The most convenient way to present these results is to construct a plot of Fj vs. x, where Fj is the force in member j due to a unit load at x, and x is taken to range over all the nodes on the bottom chord. We need to apply loads only at the nodes since the plot is linear between adjacent nodes. Plots of this type are called influence lines. Figure 2.28a shows the influence line for chord member ab. This 2.4 Influence Lines 107 Fig. 2.27 (a) Truss geometry. (b) Load position. (c) Free body diagram-transverse beam Visual representation is convenient since one can immediately identify the critical location of the load. For chord member ab, the maximum magnitude occurs when the load is applied at mid-span. Also, we note that the force is compression for all locations. Given an actual loading distribution, one evaluates the contribution of each load, and then sums the contributions. If the actual live load consisted of a uniform loading, the maximum force due to the live load would occur when the load extended over the entire span. The maximum force due to the truck is determined by positioning the truck loads as indicated in Fig. 2.28b. In general, one positions the vehicle such that the maximum vehicle load acts on node f. For chord member fg, the force is always tension. The function of the diagonal members is to transmit the vertical force from node to node along the span. This action is called "shear." The influence line for a diagonal is different in that the influence lines for upper and lower chord members, in that it has both positive and negative values. (see Fig. 2.28d vs. Fig. 2.28e). Once the influence lines are constructed, one determines the maximum value in the same way as discussed earlier. 108 2 Statically Determinate Truss Structures Fig. 2.28 (a) Load applied at node g generates compression, whereas loads at nodes f and e produce tension. Lastly, a symmetrically located diagonal such as cf has an influence line that is a rotated version of the corresponding influence line for the left symmetrically located diagonal af. Fig. 2.28 (a) Influence line for chord member ab. (b) Vehicle positioning for Fab max. (c) Influence line for chord member fg. (d) Influence line for diagonal member af. (e) Influence line for diagonal member cf. (f) Uniform unit load Because the influence lines for diagonals have both positive and negative values, one needs to consider two patterns of live load in order to establish the peak value of the member force. For member af, the extreme values are F ¼ 2=2 pffiffiffi Fmax at node f, and F ¼ 2=2 pffiffiffi at node g. For member cf, the extreme values at node e F ¼ 2=2 and at node f F ¼ 2=2. If a uniform load is applied (see Fig. 2.28f), the peak force values for both members will be: pffiffiffi Fmax ¼ 2=2 As mentioned earlier, diagonal members function to transmit vertical force to the supports. The sense of the diagonal force depends on the orientation of the diagonal. We showed above that the sense of the diagonal force depends on the orientation of the diagonal since the vertical force is positive for some positions and negative for others. ... There is important since slender members behave differently under compression vs. tension. A slender member subjected to compressive load will fail by buckling rather than by yielding since the critical buckling load is considerably less than the yield force. Therefore, from a design perspective one would avoid slender compression members. For truss type structures, this problem can be avoided by selecting an appropriate diagonal pattern. 2.4 Influence Lines 109 Fig. 2.29 Force pattern for various truss geometries. (a) Pratt truss. (b) Warren truss. (c) Howe truss As an example, consider the two diagonal patterns shown in Fig. 2.29a, b. The sense of the member forces due to a uniform live load is indicated by C (compression) and T (tension). Pattern (a) is more desirable since all the interior diagonals are in tension. However, some of the vertical members are in compression. Pattern (b) has alternating sense for the diagonals; the vertical hangers are all in tension. In general, for both truss types the top chord forces are compression and the bottom chord are tension. Figure 2.29c, d show similar results for inclined chord trusses. The designators "Pratt," "Warren," and "Howe" refer to the individuals who invented these geometric forms. Example 2.19 110 2 Statically Determinate Truss Structures Given: The structure and truck loading shown in Fig. E2.19a. Fig. E2.19a Determine: The maximum force in members ab and fg due to the truck loading. Solution: We first determine the influence lines for a unit vertical force applied along the bottom nodes. Fig. E2.19b Influence line for member ab Fig. E2.19c Influence line for member fg Then, we position the truck loading as indicated in Figs. E2.19d and E2.19e Fabmax ... Example 2.20 Influence Line Problem for a Gable Roof Structure Given: The gable roof structure shown in Fig. E2.20a. Fig. E2.20a Structural geometry and loading Determine: (i) Tabulate all the member forces due to the individual unit nodal forces applied to the top chord. We refer to this type of table as a force influence table. (ii) Use the force influence table to draw the influence lines for members cd and fg. (iii) Calculate the reactions and member forces for members cd and fg for P1 ¼ P5 ¼ 7.5 kN, and P2 ¼ P3 ¼ P4 ¼ 15 kN. 112 2 Statically Determinate Truss Structures Solution: Part (i) the member forces due to the individual unit nodal loads are listed in Fig. E2.20b. Fig. E2.20b Force results for different unit load positions The complete set of member force results are listed in the following Table. One uses this table in two ways. Firstly, scanning down a column shows the member which is most highly stressed by the loading acting at the position corresponding to the column. Scanning across a row identifies the loading which has the maximum effect on the member force. Force influence table Member ab bc cd de ef fg gh hb bi ci cd fg gd ... Part (ii) One can interpret the force influence table as representing the complete set of influence lines for the individual members. We use this data to draw influence line for member cd and fg. Fig. E2.20c Influence line for member cd Fig. E2.20d Influence line for member fgr Part (iii) By using the force influence table, the corresponding forces in members cd and fg and reactions are determined as follows: ... 2.5 Analysis of Three-Dimensional Trusses 2.5.1 Introduction Most structural systems such as highway bridges and roof systems can be considered to be composed of a set of planar trusses. However, there are exceptions, such as towers and domed structures, which cannot be decomposed into planar components and consequently one needs to deal with three-dimensional structures. These structural types are called space structures. The basic unit for a 3-D space truss is the tetrahedron, a geometrical object composed of six members that form four triangular faces. Figure 2.30 illustrates this object. We form a 3-D structure by attaching members to existing nodes. Each new node requires three members. Provided that the structure is suitably restrained with respect to rigid body motion, the displacements that the structure experiences when loaded are due only to deformation of the members. Space trusses are used for vertical structures such as towers and for horizontal long-span roof structures such as exhibition halls and covered stadiums. They usually are much more complex than simple planar trusses, and therefore more difficult to analyze. The equilibrium analysis for three-dimensional trusses is similar to that for planar trusses except that now there are three force equilibrium equations per node instead of two equations. One can apply either the method of joints or the method of sections. Manual analysis techniques are difficult to apply for large-scale structures, and one usually resorts to computer-based analysis procedures. Our immediate objectives in this section are to describe how a space structure needs to be restrained in order to prevent rigid body motion, and to illustrate some manual calculations using the methods of joints. We then present a computer-based method in the next section. 2.5.2 Restraining Rigid Body Motion A rigid three-dimensional body requires six motion restraints to be fully constrained: three with respect to translation, and three with respect to rotation. Translation restraint is provided by restraining motion in the X, Y, and Z directions. Preventing rotation is achieved by constraining motion in the X, Y, and Z directions as illustrated in Fig. 2.31. Even when suitably restrained against translation, the body can still rotate about... Fig. 2.30 Tetrahedron units for 3-D trusses 2.5 Analysis of Three-Dimensional Trusses 115 Fig. 2.31 Restraints for a 3-D rigid body Fig. 2.32 Types of supports for 3-D trusses. (a) Hinge joint. (b) Slotted roller. (c) Roller. (d) Rigid link can rotate and we need to provide additional constraints which eliminate rotation about the X, Y, and Z axes. To prevent rotation about an axis, say the X axis, one applies a translational constraint in either the Y or Z direction at a point not on the X axis. ... If one introduces more than six restraints, the structure is said to be statically indeterminate with respect to the reactions. Various examples illustrating the selection of restraints are shown (Fig. 2.32). 116 2 Statically Determinate Truss Structures Example 2.21 Various Restraint Schemes Given: The 3-D truss shown in Fig. E2.21a. Fig. E2.21 (a) 3-D truss. (b) x–y plan view. (c) x–z plan view Determine: Possible restraint schemes. Solution: The preferred way of displaying 3-D objects is to work with projections on the X–Y and X–Z planes, referred to as the "plan" and "elevation" views. The projections corresponding to the object defined in Fig. E2.21a are shown in Fig. E2.21b, c. One can employ a 3-D hinge which provides full restraint against motion in a particular direction. The "fixed" point is 3-D hinge is... the X, Y, and Z directions. Fig. E2.21d 3-D hinge at a With these restraints, the body can still rotate about either line a-b or line a-c, or a line parallel to the Z axis through a. The first two modes are controlled with Z restraints applied at b and c. The third 2.5 Analysis of Three-Dimensional Trusses 117 mode is controlled with either an X or Y restraint applied at either b or c. Figure E2.21e shows the complete set of restraints Other possible restraint schemes are shown in Figs. E2.21f, E2.21g, and E2.21h. Our strategy is to first restrain translation and then deal with the rotation modes. Fig. E2.21e Complete set of restraints Fig. E2.21f Alternative restraint scheme #1 Fig. E2.21g Alternative restraint scheme #2 Fig. E2.21h Alternative restraint scheme #3 118 2.5.3 2 Statically Determinate Truss Structures Static Determinacy The approach we followed in Sect. 2.2.2 for 2-D plane trusses is also applicable for 3-D trusses. One just has to include the additional variables associated with shifting from two to three dimensions. Each member of a truss structure has a single force measure. However, for 3-D trusses, there are three equilibrium equations per node instead of two. Defining m as the number of members, r as the number of reactions, and j as the number of nodes, the total number of force unknowns and the total number of equilibrium equations are Force unknowns ¼ m þ r Equilibrium equations ¼ 3j If m + r = 3j, the structure is said to be statically determinate. If m + r > 3j, there are more force unknowns than equations and the structure is designated as statically indeterminate. Lastly, if m + r < 3j, there are less force unknowns than required to withstand an arbitrary nodal loading, and the structure is unstable, i.e., it is incapable of supporting an arbitrary small loading. ... In addition to these criteria, the structure must be suitably restrained against rigid body motion. Example 2.22 A Stable Determinate Truss Given: The truss defined in Fig. E2.22a, b. 2.5 Analysis of Three-Dimensional Trusses 119 Fig. E2.22 (a) 3-D truss. (b) x–y plan view. Determine: The stability. Solution: For this structure, there are eight members, seven reactions, and five joints. ... The structure is statically determinate. Example 2.23 An Unstable Structure Given: The truss defined in Figs. E2.23a and E2.23b Determine: The stability. Solution: There are six reactions and they intersect at a common point, c0 , shown in Fig. E2.23a. As a result, the structure cannot resist rotation about the Z axis through c0 . 120 2 Statically Determinate Truss Structures Fig. E2.23a x–y plan view Fig. E2.23b x–z plan view 2.5.4 Method of Joints for 3-D Trusses Each member of a space truss is assumed to be pinned at its ends to nodes in such a way that there is no bending in the member, only an axial force whose direction coincides with the centroidal axis. The direction of the force is governed by the geometry of the member, so one needs only to determine the magnitude. We find these quantities using force equilibrium equations associated with the global equilibrium conditions. Once the reactions are known, we 2.5 Analysis of Three-Dimensional Trusses 121 Fig. 2.33 Resolution of a force into its components range over the nodes and establish the nodal force equilibrium equations. This process is similar to the method of joints for planar Trusses except that now there are three equations per node. The member forces are resolved into X, Y, and Z components. ... where the direction cosines are expressed in terms of the member lengths and projections ... Example 2.24 Analysis of a Tripod Structure Given: The structure shown in Fig. E2.24a, b. ... Determine: The force in each member. Fig. E2.24 Tripod geometry and supports. (a) x–y plan view. (b) x–z plan view 2.5 Analysis of Three-Dimensional Trusses 123 Solution: There are three reactions per support, making a total of nine reaction forces. Adding the three unknown member forces raises the total number of force unknowns to 12. Each joint has three force equilibrium equations and there are four joints, so the structure is statically determinate. ... Example 2.25 Tetrahedron Analysis Given: The structure and loading shown in Fig. E2.25a, b. Determine: The member forces. ... Example 2.26 Computation of Displacements – 3-D Truss Given: The truss defined in Fig. E2.26a. Determine: The displacement in the X direction at joint d. ... 2.6 Matrix Formulation: Equilibrium Analysis of Statically Determinate 3-D Trusses ... The analysis approaches described in the previous sections can be automated for computer-based analysis. In what follows, we describe a matrix-based formulation for the equilibrium analysis of statically determinate 3-D trusses. We present a general matrix formulation later in Chap. 12. 2.6.1 Notation A truss is an assembly of nodes that are interconnected with members. It is convenient to define the geometry with respect to a global Cartesian coordinate system, XYZ, and number the nodes and members consecutively. Figure 2.34 illustrates this scheme. The structure has four nodes and six members. We assume a positive sense for each member and define the direction cosines consistent with the assumed sense. The positive and negative nodes for member n are denoted as n+ and n−. ... 2.6.2 Member–Node Incidence The direction cosines can be assembled into an array ... 2.6.3 Force Equilibrium Equations ... 2.6.4 Stability A structure is said to be stable when a unique solution for the member forces exists for a given set of external loads. The relationship between the loading and the resulting member forces is defined by the linear matrix equation, (2.27). Noting Cramer's rule [4], the stability requirement can be expressed as determinant ... 2.6.5 Matrix Formulation: Computation of Displacements The Method of Virtual Forces ... 2.7 Summary 2.7.1 Objectives of the Chapter • To develop a criteria for assessing the initial stability of truss type structures • To present methods for computing the axial forces in the members of statically determinate trusses • To present a matrix-based formulation for the analyses of arbitrary statically determinate trusses • To present methods for computing the displaced configuration of a truss • To introduce the concept of an influence line and illustrate its application to trusses 142 2.7.2 2 Statically Determinate Truss Structures Key Facts and Concepts • The statical determinacy of a planar truss depends on the number of unknown forces vs. the number of available force equilibrium equations. • The forces in the members of a statically determinate truss are independent of the member properties such as area and material modulus and support movements. • The two force analysis procedures are the method of joints and the method of sections. The method of joints works with a single node. Applying the procedure to every node leads to the joint member forces. The method of sections is designed to allow one to determine the force in a particular member. One passes a cutting plane through the structure, selects either segment, and applies the equilibrium conditions. This method requires less computation and generally is easier to apply. • Given the external loads, one can determine the internal member forces using force equilibrium equations when the truss is statically determinate. The displacements due to the loading can be computed manually using the method of virtual forces. To determine the displacement at a point A in a particular direction, da, one applies a virtual force δPA at point A in the same direction as the desired displacement and computes, using static equilibrium equations, the internal forces δF, and reactions, δR, due to δPa. The displacement is given by X X da δPa ¼ e δF d δR members reactions where d is the prescribed support movement and e is the elongation of the member due to force, temperature change, and initial fabrication error. FL e¼ þ αL ΔT Lþ e0 AE This method is restricted to static loading and small displacements. It is also applicable for statically indeterminate trusses when the member forces are known. • The concept of influence lines is very useful for dealing with the live load which can act anywhere on the structure. Given a particular member force and a particular type of live loading, usually a unit vertical loading, the influence line displays graphically the magnitude of the force for various locations of the load. By viewing the plot, one can immediately determine the position of the load that produces the peak magnitude of the member force. 2.8 Problems Classify each of the following plane trusses defined in Problems 2.1–2.4 as initially stable or unstable. If stable, then classify them as statically determinate or indeterminate. For indeterminate trusses, determine the degree of static indeterminacy. 2.8 Problems Problem 2.1 Problem 2.2 143 144 2 Statically Determinate Truss Structures Problem 2.3 Problem 2.4 Determine all the member forces for the plane trusses defined in Problems 2.5–2.12 using the method of joints. Problem 2.5 2.8 Problems Problem 2.6 Problem 2.7 145 146 Problem 2.8 Problem 2.9 Problem 2.10 2 Statically Determinate Truss Structures 2.8 Problems 147 Problem 2.11 Problem 2.12 Determine all the member forces for the plane trusses defined in Problems 2.13–2.16 using a combination of the method of joints and the method of sections. 148 Problem 2.13 Problem 2.14 Problem 2.15 2 Statically Determinate Truss Structures 2.8 Problems Problem 2.16 Problem 2.17 Problem 2.18 149 150 2 Statically Determinate Truss Structures 2.8 Problems Problem 2.19 Use the principle of virtual forces to determine the vertical displacement at joint b due to loading shown and temperature increase of ΔT ¼ 40 F for members ab and bc. Assume A ¼ 1.4 in.2, E ¼ 29,000 ksi, and α ¼ 6.5 (106)/ F Problem 2.20 Use the principle of virtual forces to determine the horizontal displacement at joint c. E ¼ 200 GPa. The areas of the members are as follow: Aab ¼ Abc ¼ Abe ¼ 1290 mm2 Abf ¼ Abd ¼ 645 mm2 Acd ¼ Ade ¼ 1935 mm2 Aaf ¼ Afe ¼ 2580 mm2 2.8 Problems 151 Problem 2.21 For the plane truss shown below, use the principle of virtual forces to determine the vertical displacement at joint C due to the loading shown and a settlement of 0.5 inch at support a. Assume A ¼ 2 in.2 and E ¼ 29,000 ksi. Problem 2.22 For the plane truss shown below, use the principle of virtual forces to determine the vertical and horizontal displacement at joint d. A ¼ 1300 mm2 E ¼ 200 GPa Problem 2.23 Use the principle of virtual forces to determine the horizontal and vertical displacement at joint b due to: (a) Loading shown. (b) Temperature increase of ΔT ¼ 16 C for members ab and bc. A ¼ 900 mm2 E ¼ 200 GPa α ¼ 12 106 = C 152 2 Statically Determinate Truss Structures 2.8 Problems Problem 2.24 Use the principle of virtual forces to determine the horizontal component of the displacement at joint e. Assume A ¼ 0.5 in.2 and E ¼ 29,000 ksi. (i) For the loading shown (ii) For a fabrication error of 0.25 in. for members ac and df (iii) For the summation of Case (i) and Case (ii) loadings. Problem 2.25 Use the principle of virtual forces method to determine the horizontal component of the displacement at joint b. Assume A ¼ 0.5 in.2, E ¼ 30,000 ksi, α ¼ 6.5 106/ F (i) For the loading shown (ii) For a temperature increase of ΔT ¼ 60 F for all members (iii) For the summation of Case (i) and Case (ii) loadings. 2.8 Problems 153 Problem 2.26 For the plane truss shown below, use the principle of virtual forces to determine the vertical displacement at joint f. A ¼ 2 in.2, E ¼ 29,000 ksi Problem 2.27 For the plane truss shown below, determine the required cross-sectional area for the bottom chord members to limit the vertical deflection at d to 0.56 in. Assume equal cross-sectional areas. E ¼ 29,000 ksi. 154 2 Statically Determinate Truss Structures 2.8 Problems Problem 2.28 For the plane truss shown in Problem 2.12, use the principle of virtual forces to determine the vertical displacement at joint g. The areas are 4 in.2 for top chord members, 3 in.2 for bottom chord members, and 2 in.2 for other members. E ¼ 29,000 ksi. Problem 2.29 Suppose the top chord members in the truss defined above experience a temperature decrease of 60 F. Determine the resulting displacements, u and v. A ¼ 6 in.2, E ¼ 29,000 ksi, α ¼ 6.5 10-6/ F. Problem 2.30 Solve Problem 2.15 using computer software. Assume the cross-sectional areas are equal to A. (a) Demonstrate that the member forces are independent of A by generating solutions for different values of A. (b) Determine the value of A required to limit the vertical displacement to 50 mm. Problem 2.31 Consider the complex truss defined below in Figure (a). Use computer software to determine the member forces for the loading shown in Figure (a). (a) Assume equal areas (b) Take an arbitrary set of areas (c) Determine the member forces corresponding to the loading shown in Figure (b). Are the forces similar to the results of part (a). Discuss. 2.8 Problems 155 Problem 2.32 Solve Problem 2.11(a) using computer software. Assuming the cross-sectional areas are equal to A. Demonstrate that the member forces are independent of A by generating a solution of different values of A. Determine the area for which the maximum displacement is 30 mm. E ¼ 200 GPa. 156 2 Statically Determinate Truss Structures Problem 2.33 Consider the complex truss defined below. Assume equal areas. Use computer software to determine the member forces. 2.8 Problems 157 Problem 2.34 For the truss and the loading shown: (a) Tabulate all the member forces due to the individual unit vertical load forces applied to the top chord (force influence table). Use computer software. (b) Use the force influence table in part (a) to (i) Draw influence lines for members 15, 4, and 20. (ii) Calculate the member forces in members 3, 19, 10, and 14 for the following loading: P2 ¼ 10 kN, P4 ¼ 6 kN, and P6 ¼ 8 kN. Problem 2.35 For the truss and the loading shown: (a) Tabulate all the member forces due to the individual unit vertical loads applied to the bottom chord (force influence table). Use computer software. (b) Use the force influence table in part (a) to (i) Draw influence lines for members bc, cm, and ji. (ii) Calculate the member forces in members ke, de, kl, and ei for the loading P1 ¼ 8 kip, P3 ¼ 10 kip, and P4 ¼ 4 kip. Problem 2.36 The roof structure shown below consists of trusses spaced uniformly, 20 ft (6 m) on center, along the length of the building and tied together by purlins and x-bracing. The roofing materials are supported by purlins which span between trusses at the truss joints. 158 2 Statically Determinate Truss Structures Consider the following loadings: Dead load: roof material, purlins, truss members, estimated at 15 psf (720 Pa) of roof surface Snow load: 30 psf (960 Pa) of horizontal projection of the roof surface Wind load: windward face 12 psf (575 Pa), leeward face 8 psf (385 Pa) normal to roof surface Determine the following quantities for the typical interior truss: (a) Compute the truss nodal loads associated with gravity, snow, and wind. (b) Use computer software to determine the member forces due to dead load, snow load, and wind load. Tabulate the member force results. 2.8 Problems Problem 2.37 Determine the member forces for the space truss shown. Problem 2.38 Determine the member forces for the space truss shown. 159 160 2 Statically Determinate Truss Structures Problem 2.39 Determine the member forces for the space truss shown. References 161 Problem 2.40 Determine the member forces for the space truss shown. Problem 2.41 For the space truss shown in Problem 2.37, use the principle of virtual forces to determine the displacements u, v, and w at joint d. E ¼ 29,000 ksi, and A ¼ 3.0 in.2 References 1. Faraji S, Ting J, Crovo DS, Ernst H. Nonlinear analysis of integral bridges: finite element model. ASCE J Geotech Geoenviron Eng. 2001;127(5):454–61. 2. Mathcad 14.0, Engineering calculations software. 3. Tauchert TR. Energy principles in structural mechanics. New York: McGraw-Hill; 1974. 4. Connor JJ. Introduction to structural motion control. Upper Saddle River: Prentice Hall; 2003. 3 Statically Determinate Beams Abstract Our focus in this chapter is on describing how beams behave under transverse loading, i.e., when the loads act normal to the longitudinal axes. This problem is called the "beam bending" problem. The first step in the analysis of a statically determinate beam is the determination of the reactions. Given the reactions, one can establish the internal forces using equilibrium-based procedures. These forces generate deformations that cause the beam to displace. We discuss in detail the relationship between the internal forces and the corresponding displacements and describe two quantitative analysis procedures for establishing the displacements due to a particular loading. The last section of the chapter presents some basic analysis strategies employed in the design of beams such as influence lines and global envelopes. 3.1 Definition of a Prismatic Beam Beams are used extensively in structures, primarily in flooring systems for buildings and bridges. They belong to the line element category, i.e., their longitudinal dimension is large in comparison to their cross-sectional dimensions. Whereas truss members are loaded axially, beams are loaded normal to the longitudinal direction, and carry the loading by bending and twisting action. This mode is illustrated in Fig. 3.1. The transverse loading produces transverse deflection, which results in a nonuniform distribution of stress throughout the body. Most of the applications of beams in building structures involve straight beams with constant cross-sections. We refer to this subgroup as prismatic beams. Figure 3.2 defines the geometrical parameters and notation used for prismatic beams. The longitudinal axis-X passes through the centroid of the cross-section, and the Y, Z axes are taken as the principal inertia directions. The relevant definition equations are # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_3 163 164 3 Statically Determinate Beams Fig. 3.1 Beam crosssections and bending mode. (a) Simple supported beam. (b) Section A—A—crosssection examples. Rectangular, T shape, I shape. (c) Bending mode ð ð ð y dA ¼ A z dA ¼ A ð A Iz ¼ y2 dA Iy ¼ z2 dA ð3:1Þ z2 dA A These conditions ensure that when the applied loads are in the X Y plane, points on the longitudinal axis will not displace in the Z direction. Figure 3.3 illustrates this mode of behavior, the longitudinal axis-X becomes a curve v(x) contained in the X Y plane. This type of behavior is Stabilitycalled planar bending. There are cases where the line of action of the loading does not pass through the X-axis, such as illustrated in Fig. 3.4. The eccentricity produces a torsional moment about the X-axis, and the crosssection will rotate as well as deflect. This behavior is called "combined bending and torsion." A "prismatic member acted upon by just a torsional moment will experience only torsional behavior, i.e., the cross-section will just twist. Mechanics of Solids texts deal with stresses and strains in beams. Our objective here is not to redevelop this material but rather to utilize it and formulate a structural theory for beams that will provide the basis for analyzing the behavior of structures composed of beam elements. Since structural theory is founded on Engineering Mechanics Theory, at least one subject dealing with 3.2 Stability and Determinacy of Beams: Planar Bending 165 Fig. 3.2 Notations for prismatic beam— symmetrical cross-section Engineering Mechanics is usually required before studying Structural Theory. We assume that the reader has this level of exposure to Engineering Mechanics. 3.2 Stability and Determinacy of Beams: Planar Bending We presented the general concept of stability of a rigid body in Chap. 1 and used the general concept to develop stability criteria for truss-type structures in Chap. 2. In what follows, we examine the stability question for beam-type structures and develop similar criteria. For completeness, we first briefly review the basis for stability discussed in Chap. 1. Consider the rigid body shown in Fig. 3.5. The body can move only in the X Y plane. There are two types of planar motion for a rigid body: translation in the x direction, uA, translation in the y direction, vA, and rotation about an axis normal to the X Y plane, ωA. A body is said to be stable when rigid body motion is prevented. Therefore, it follows that one must provide three motion constraints to restrain motion in the X Y plane. 166 3 Statically Determinate Beams Fig. 3.3 Planar bending mode Fig. 3.4 Combined bending and torsion One needs to be careful in selecting the orientation of the three translation constraints. Consider Fig. 3.6. We first choose two directions, "a" and "b" in the X Y plane. They intersect at point o. With these two constraints, the only possible rigid body motion is rotation about point o. If we take the third direction as "c," this rotation is not prevented. Therefore, it follows that the three directions must be nonconcurrent as well as coplanar, i.e., they cannot intersect at a common point. This implies that they must not be parallel. Any other direction, such as "d0 " is permissible. 3.2 Stability and Determinacy of Beams: Planar Bending 167 Fig. 3.5 Planar rigid body motions Fig. 3.6 Concurrent displacement constraints When the loading is arbitrary, the body needs to be constrained against motion in any plane. This requires six constraints, three with respect to translation and three with respect to rotation about the X, Y, and Z directions. The strategy for selecting restraints is similar to the treatment of 3-D truss structures. We point out that for pure rotational loading only one rotational restraint is required. 168 3 Statically Determinate Beams Motion constraints produce reaction forces when the body is loaded. The nature of the reaction forces depends on the constraints. Various types of supports for beams subjected to planar bending are illustrated below. 3.2.1 Fixed Support: Planar Loading The beam is embedded at point A in such a way that the end is prevented from translating or rotating. We say the member is "fixed" at A. The reactions consist of two forces and one moment. 3.2.2 Hinged Support: Planar Loading Suppose A is to be fully restrained against translation. This can be achieved by pinning the member. Horizontal and vertical restraints are provided and the reactions are two forces. 3.2 Stability and Determinacy of Beams: Planar Bending 3.2.3 169 Roller Support: Planar Loading Suppose A is to be restrained against motion perpendicular to the surface of contact. We add a restraint to A by inserting a device that allows motion parallel to the surface of contact but fully restrains motion in the direction perpendicular to the surface. We refer to this device as a roller. This restraint produces a reaction force perpendicular to the surface of contact. When the loading is three dimensional, additional restraints are required. The supports described above needs to be modified to deal with these additional restraints. Typical schemes are shown below. 170 3 3.2.4 3-D Fixed Support 3.2.5 3-D Hinged Support 3.2.6 3-D Roller Support: Z Direction Statically Determinate Beams 3.2 Stability and Determinacy of Beams: Planar Bending 3.2.7 171 Static Determinacy: Planar Beam Systems In general, a body restrained with three nonconcurrent coplanar displacement constraints is stable for planar loading. When loading is applied, the only motion that occurs is due to deformation of the body resulting from the stresses introduced in the body by the loading. The motion restraints introduce reaction forces. Since there are three equations of force equilibrium for a body, and only three unknown forces, one can determine these unknowns using only the force equilibrium equations. In this case, we say that the structure is stable and statically determinate. If a body is over restrained, i.e., if there are more than three nonconcurrent displacement restraints, we say that the structure is statically indeterminate. This terminology follows from the fact that now there are more than three force unknowns and consequently one cannot uniquely determine these unknowns with only the three available force equilibrium equations. Statically indeterminate structures require a more rigorous structural theory and therefore we postpone their treatment to part II of the text. In what follows, we present some examples of statically determinate and statically indeterminate beams. 3.2.8 Unstable Support Arrangements The beam shown above has the proper number of constraints, but they all vertical. There is no constraint against horizontal motion, and therefore the beam is unstable. The roller support at B constrains motion in the horizontal direction but does not prevent rigid body motion about point A. 3.2.9 Beam with Multiple Supports Three are vertical restraints and one horizontal restraint. These restraints produce the four reaction forces shown below. 172 3 Statically Determinate Beams Fig. 3.7 Two-span beam Fig. 3.8 Beam with moment release Fig. 3.9 Free body diagram for beam with moment release One of the vertical restraints is redundant, i.e., is not needed for stability and therefore can be deleted. Deleting the support at B results in the structure shown below. A beam supported only at its ends in a minimal way is referred to as a simple supported beam. The beam depicted in Fig. 3.7 is called a two-span continuous beam. This beam is statically indeterminate to the first degree. We will show later that multi-span continuous beams are more structurally efficient than simply supported beams in the sense that they deflect less for a given design loading. 3.2.10 Beam with a Moment Release Suppose we cut the beam shown in Fig. 3.8 at point D and insert a frictionless hinge. We refer to the hinge as a moment release since the moment is zero. The hinge does not restrain rotation at D, and member DC is free to rotate about D. The beam is now statically determinate. The corresponding reaction forces are listed below on the free body diagrams (Fig. 3.9). 3.3 Reactions: Planar Loading 173 Fig. 3.10 Three-span beam Fig. 3.11 Statically determinate versions of three-span beam with moment releases Member DC is statically determinate since there are only three reaction forces. Once the forces at D are known, the remaining reactions for member ABD can be determined. Therefore, it follows that inserting a hinge at D reduces the degree of indeterminacy by 1 . We consider next the three-span continuous beam shown in Fig. 3.10. This structure is indeterminate to the second degree since there are two extra vertical supports. One can reduce the structure to a statically determinate structure by inserting two moment releases. Various possibilities are listed in Fig. 3.11. The optimal location of moment releases is illustrated in Examples 3.33 and 3.34. 3.3 Reactions: Planar Loading When a structure is subjected to external loads, the displacement restraints develop reaction forces to resist the tendency for motion. For structures that are statically determinate, we can determine these forces using the three global force equilibrium equations for planar loading applied to a body. One selects a set of directions–n and s–s, where s–s is not parallel to n–n. The steps are ∑Þ Summation of forces in direction n ¼ 0 ∑Þ Summation of forces in direction s ¼ 0 where direction s is not parallel to direction n ∑Þ Summation of moments about an arbitrary point, A ¼ 0 ð3:2Þ 174 3 Statically Determinate Beams One constructs a free body diagram of the structure and applies these equations in such a way as to obtain a set of uncoupled equations, which can be easily solved. When a statically indeterminate structure has a sufficient number of releases such that it is reduced to being statically determinate, we proceed in a similar way except that now we need to consider more than one free body. The following series of examples illustrate the strategy for computing the reactions. Example 3.1 Beam with Two Over Hangs Given: The beam shown in Fig. E3.1a. Determine: The reactions. Fig. E3.1a Solution: Summing moments about B leads to the vertical reaction at A. X MB ¼ 0 1 2 8 2 RA ð20Þ þ 18 ð10Þ ¼ 0 ð2 3 :RA ¼ 4:3 " Summing the vertical forces, 3.3 Reactions: Planar Loading 175 X FY ¼ 0 ∴RB ¼ 5.3 " X RB þ 4:3 1 2 8 2 þ 18 ¼ 0 ð20ð Fig. E3.1b Summing the vertical and horizontal forces, X FX ¼ 0 ∴RB ¼ 0 Example 3.2 Simply Supported Beam Given: The beam shown in Fig. E3.2a. Determine: The reactions. Fig. E3.2a Solution: As a first step, we construct the free body diagram for the beam. The reaction at B is normal to the inclined surface. We resolve it into horizontal and vertical components using (Fig. E3.2b) Fig. E3.2b RBy ¼ RB cos α RBx ¼ RB sin α Summing moments about A leads to the vertical reaction at B. 176 3 Statically Determinate Beams X MA ¼ 0 L ¼ 0 ∴RBy ¼ wL 2 L RBy wL ¼ 2 Given RBy, we determine RB and RBx, and finally the reaction RB RB ¼ RBy wL ¼ 2 cos α cos α The horizontal component is RBx ¼ RB sin α ¼wL tan α ← 2 We determine the reactions at A using force summations. X X Fx ¼ 0 Fy ¼ 0"þ RAx ¼ RBx RAy ¼ wL tan α ! 2 wL ¼ RBy ¼ " 2 When α ¼ 0 the horizontal reaction vanishes. Suppose w ¼ 30 kN/m, α ¼ 30 and L ¼ 10 m. The reactions are listed below. Example 3.3 Two-Span Beam with a Moment Release Given: The beam shown in Fig. E3.3a. Determine: The reactions. Fig. E3.3a 3.3 Reactions: Planar Loading 177 Determine: The reactions. Fig. E3.3a Solution: The most direct way of analyzing this structure is to first work with a free body diagram of beam segment DC. Applying the equilibrium conditions to this segment results in X X X MD ¼ 0 FY ¼ 0 Fx ¼ 0 Fx ¼ 0 " 4 wL " VD ¼ 4 RC ¼ HD ¼ 0 With the internal forces at D known, we can proceed with the analysis of segment ABD. 178 3 Statically Determinate Beams Summing moments about A leads to the vertical reaction at B. X X RB X MA ¼ 0 ð75LÞð0:5wLÞ ð1:5LÞ0:25wLÞ LRB ¼ 0 ∴RB ¼ 1:5wL " Summing the vertical and horizontal forces, X FY ¼ 0 RAy ¼ 1:75wL ∴RAy ¼ 0:25wL↓ X Fx ¼ 0 ∴RAx ¼ 0 The reactions are listed below. If the hinge was placed at point B, the structure would behave as two simply supported beams, as shown below. 3.3 Reactions: Planar Loading 179 Example 3.4 Three-Span Beam Given: The beam shown in Fig. E3.4a. Determine: The reactions. Fig. E3.4a Solution: Summing moments about D leads to the vertical reaction at C. X MD ¼ 0 X MB ¼ 0 32Þ 18 1503Þ07:5Þ ¼ 0 ∴RD ¼ 43:25 " RD ð6Þ þ 28 2 Summing the vertical forces, X FY ¼ 0 RB þ 43:25 1503Þ 0 ∴RB ¼ 31:75 " The reactions are listed below. 180 3 Statically Determinate Beams Example 3.5 Three-Span Beam with Two Moment Releases Given: The beam shown in Fig. E3.5a. Determine: The reactions. Fig. E3.5a 3 ¼0 2 180 3 Statically Determinate Beams Solution: We first work with a free body diagram of beam segment EF. Then, with the internal forces at E and F known, we proceed with the analysis of segment ABE and FCD. The reactions are listed below. Example 3.6 Horizontal Beam Supporting a Vertical Sign Given: The structure defined in Fig. E3.6a. Determine: The reactions at A and C. Fig. E3.6a Member BED is rigidly attached to the beam, ABC. Member FG is simply supported on member BED. Assume the force in member BED. Even though FG is self-weight, W and is acted upon by a uniform horizontal wind load p. This structure is an idealization of a highway sign supported on a beam. 3.3 Reactions: Planar Loading 181 Determine: The reactions. Fig. E3.6a Solution: We work with two free body diagrams, one for member FG and the other for the remaining part of the structure. Consider first member FG. Enforcing equilibrium leads to: VF ¼ W HF ¼ HG ¼ ph 2 Next, we apply these forces to the structure composed of member ABC and member BED. The free body diagram is shown below. 182 3 Statically Determinate Beams Summing moments about A leads to RC X MA ¼ 0 L ph ph W ¼ ð2hÞ þ ð2:5hÞ þ RC L 2 4 4 W h ∴RC ¼ ph 1:125 2 L The horizontal and vertical reactions at A are ph 2 W h RAy ¼ ph 1:125 2 L RAx ¼ Note that the vertical reaction at C may become negative if ph is large with respect to W and h is of the order of L. 3.4 Internal Forces: Planar Loading We have shown that external loads are resisted by reaction forces at the supports. The question we now need to address is: What is the effect of this combination of external loads and reaction forces on the body? We answer this question by examining the equilibrium of an arbitrary segment of the beam. Consider the uniformly loaded, simply supported beam shown in Fig. 3.12a. We pass a cutting plane a distance x from the left end and consider either the left or right segment. The external loads create a force unbalance. To maintain equilibrium, a vertical force, V(x), and a moment, M(x), are required at the section. We refer to these quantities as the internal shear force and bending moment. The magnitudes of V(x) and M(x) for this section are found by applying the equilibrium conditions to the segment. 3.4 Internal Forces: Planar Loading 183 Fig. 3.12 Internal forces. (a) Segmented beam. Fig. 3.13 Sign convention for internal forces wL þ w x wL ¼ 2 VðxÞ¼ wL x 2 M ðxÞ ¼ x wL 2 þ wx 2 x 2 þ ðxÞ ¼ w x ðL xÞ 2 ð3:3Þ We need to first define a sign convention for the positive directions of the internal force quantities. This is shown in Fig. 3.13 for a positive face, i.e., a face whose outward normal points in the + X direction. The shear force is positive when it points in the + Y direction, and the positive sense for moment is from X to Y. Depending on the external loading, there may also be an axial force. The positive sense for the axial force is taken as the + X direction. These directions are reversed for a negative face. This sign convention is also used in the matrix formulation of the beam bending problem which is the basis for computer-based analysis software. Historically, some authors use a sign convention for shear which is opposite to this convention. We prefer to employ the above convention since it is consistent with the output of structural software systems and therefore allows the reader to transition easily from analytical to digital computation schemes. 184 3 Statically Determinate Beams Fig. 3.14 Shear and moment diagrams Figure 3.14 shows the variation of these quantities for the beam defined in Fig. 3.12. The shear varies linearly, with maximum values at the support. The moment varies parabolically, and the maximum value occurs at mid-span. These plots are called "shear" and "moment" diagrams. Positive moment is plotted on the top face in the USA. In the UK, positive moment is plotted on the bottom face. Again, it is a question of what convention one is most comfortable with. The maximum bending moment and shear force are used to determine the dimensions of the crosssection. The specific design procedure depends on the material selected, such as wood, steel, or concrete, and the design code adopted. One constructs the internal force distributions by selecting various cutting planes, evaluating the corresponding values, and then extrapolating between the sections. With some experience, one can become very proficient at this operation. We illustrate the process with the following examples. Example 3.7 Cantilever Beam with Multiple Concentrated Loads Given: The cantilever beam with two concentrated loads shown in Fig. E3.7a. 3.4 Internal Forces: Planar Loading 185 Determine: The shear and moment diagrams. Solution: We start by determining the reactions at A by enforcing the equilibrium equations. X Fy ¼ RA 20 40 ¼ 0) RA ¼ 60 kN X MA ¼ 20ð3Þ 40ð6Þ þ MA ¼ 0) MA ¼ 300 kN m Then we pass a cutting plane between A and B for x < 3 V ðxÞ ¼ 60 MðxÞ ¼ 60x 186 3 Statically Determinate Beams Lastly, we cut between B and C. 3 L/2 L/2 PL2 θmax ¼ 16EI PL3 ¼ 48EI vmax L ¼ ... PL3 ... vmax ¼ ... 48EI 2208 ... 3. Uniform Loading (Fig. E3.20d) Fig. E3.20d The equations for M(x), θ(x), and v(x) for a uniform load are as follows: w ... Rotations of the ends ... wL3 24EI ... 5wL4 384EI wL3 24EI at x¼L 2 Note that the rotation is zero at mid-span since the structure and the loading are symmetrical. Table 3.1 catalogs the results of Example 3.20. 3.5 Deformation-Displacement Relations ... Table 3.1 (continued) M* L M* x0B ¼ 6EI PL2 2EI 0 θ ... Rotation θ+ counter clockwise ... Translation ... 3.6 Displacement and Deformation of Slender Beams 3.6.4 231 Computing Displacements with the Method of Virtual Forces The procedures described in the previous section are intended to generate analytical solutions for the displacement and rotation. In many cases, one is interested only in the motion measures for a particular point, and it is more convenient to determine only these quantities at the point of interest, one can apply the Method of Virtual Forces. We described in detail the Method of Virtual Forces specialized for beams with slender, which is presented in [1]. We express the principle as δ dP ¼ ∫ δbending deformation δM δM dx (3.35) where δP is the virtual force in the direction of d, and δM(x) is the virtual moment corresponding to δP. The deformation due to transverse shear is not included since it is negligible for slender beams. When the behavior is linear elastic, the bending deformation is related to the moment by bending deformation dx ¼ M/EI dx ... The steps involved in applying the method of virtual forces are as follows: We use as an example the beam shown in Fig. 3.28. To determine a desired vertical displacement or rotation, such as vA or θB, one applies the corresponding virtual force or virtual moment in the direction of the desired displacement or rotation, determines the virtual moment δMv(x) or δMθ(x), and then evaluates the following integrals. ... Fig. 3.28 Actual and virtual loads. (a) Actual load M(x). (b) Deflected shape. (c) Virtual load δMv for vA. (d) Virtual load δMθ(x) for θB. 232 3 Statically Determinate Beams Just as we did for truss structures in Chap. 2, one can take δP to be a unit force. The application of this procedure is illustrated in the following examples. Example 3.21 Deflection Computation—Method of Virtual Forces Given: A uniformly distributed cantilever beam shown in Fig. E3.21a. Fig. E3.21a Determine: The vertical deflection and rotation at B. Take EI as constant. Solution: We first determine the moment distribution corresponding to the applied loading. This is defined in Fig. E3.21b. The virtual moment distributions corresponding to vB, θB are defined in Figs. E3.21c and E3.21d. Note that we take δP to be either a unit force or a unit moment (for rotation). Fig. E3.21b M(x) The actual moment M(x) is ... Vertical deflection at B: We apply the virtual force δP ¼ 1 at point B and compute the corresponding virtual moment. ... 3.6 Displacement and Deformation of Slender Beams Fig. E3.21c δMvB δxÞ Then, noting (3.36) ... Integrating leads to vB ¼ wL4 # 8EI Rotation at B: We apply the virtual moment δP ¼ 1 at point B and determine δM(x). This loading produces a constant bending moment, 0≤L δMθB δxÞ ¼ 1 Fig. E3.21d δMθB δxÞ Then, noting (3.36) ... Finally, one obtains θB ¼ wL3 clockwise 6EI 233 234 3 Statically Determinate Beams Fig. E3.22a Example 3.22 Deflection Computation—Method of Virtual Forces Given: The simply supported beam shown in Fig. E3.22a. Determine: The vertical deflection at point C located at mid-span. Take EI as constant. Solution: The moment distribution corresponding to the applied loading is defined in Fig. E3.22b. The virtual moment distributions corresponding to vC are defined in Figs. E3.22c and E3.22d. Note that we take δP to be either a unit force or a unit moment (for rotation). Fig. E3.22b M(x) ... 3.6 Displacement and Deformation of Slender Beams Vertical displacement at C: We apply a unit load at point C and determine δM(x). Fig. E3.22c δMvC(x) ... 260 3 Statically Determinate Beams ... Letting x range from 0 to L leads to the plot shown in Fig. 3.50c. The maximum value of MA occurs when the load is acting at point A. ... max ... This value provides input for the moment envelope. We repeat the computation taking different points such as xA, xB, and xc. The conventional way of representing this data is to show the discrete points along the span and list the corresponding absolute values as shown in Fig. 3.50d. Figure 3.50d illustrates this approach. 260 3 Influence Lines and Force Envelopes for Statically Determinate Beams Fig. 3.50 (a) Beam. (b) Loading patterns – Concentrated load P at x < xA and x > xA. (c) Moment diagram. (f) Different load patterns. (e) Moment diagram. (f) Shear diagrams for concentrated load P at x < xA and x > xA. (g) Influence line for shear at location xA. (h) Absolute maximum moment. (i) Shear force envelope 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 261 Fig. 3.50 (continued) Location xA : xB : xc Maximum positive moment. ... One evaluates these expressions for a sufficient number of points so that the local extremities are identified. The limiting form of the moment envelope based on many points is a parabola. We proceed in a similar manner for the shear force envelope. The shear diagram for a single concentrated force applied at x is shown in Fig. 3.50f. 262 3 Statically Determinate Beams Suppose we want the influence line for the shear at location xA. Noting Fig. 3.50f, the shear force at xA for the different positions of the load is x < xA x > xA ... These functions are plotted in Fig. 3.50g. At point xA, there is a discontinuity in the magnitude of V equal to P and a reversal in the sense. This behavior is characteristic of concentrated forces. To construct the force envelope, we note that maximum and minimum values of shear at point A are V A max ¼ ... These values are plotted on the span at point A (Fig. 3.50h). Repeating the process for different points, one obtains the force envelope shown in Fig. 3.50i. Example 3.25 Construction of Influence Lines Given: The beam shown in Fig. E3.25a. Fig. E3.25a Determine: The influence lines for the vertical reaction at B and C, the moment and shear at section 1-1. Suppose a uniformly distributed live load of w ¼ 1.2 kip/ft and uniformly distributed dead load of wD ¼ 0.5 kip/ft are placed on the beam. Using these results, determine the maximum value of the vertical reaction at B and the maximum and minimum values of moment at section 2-2. Solution: Note that the influence lines are linear because the equilibrium equations are linear in the position variable (see Fig. E3.25b). 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams Fig. E3.25b 263 264 3 Statically Determinate Beams Fig. E3.25d Maximum and minimum values of the moment. The peak value of the reaction at B is determined by loading the beam as indicated in Fig. E3.25c. ... The peak values of moment at section 2-2 are generated by using the loadings shown in Fig. E3.25d. ... 265 Fig. E3.25c Influence lines for RB, RC, V1-1, M2-2, and M1-1 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 265 The influence lines for the reactions ... 266 3 Statically Determinate Beams Fig. E3.25e ... Example 3.26 Given: The two-span beam shown in Fig. E3.26a. There is a hinge (moment release) at the center span. Determine: The influence line for the moment at E and the moment envelope for the beam. Fig. E3.26a Solution: The influence lines for the bending moment at E and the moment envelope are plotted in Fig. E3.26b. Figure E3.26b shows the influence line for the vertical reaction at A. The influence line for moment at D equals zero when the load is applied to member ABC. x RD ¼ 1:5 þ L x RB ¼ 3 2 for 1:5L < x < 2L L x 2 RA ¼ L 268 3 Statically Determinate Beams The moment distribution corresponding to these loading cases are plotted in Fig. E3.26c. Fig. E3.26c We note that the moment at E is positive when the load is on span AB, and switches to a negative value when the load moves on to span BCD. The influence line for the bending moment at E is plotted in Fig. E3.26d. Fig. E3.26d The moment envelope is constructed using Fig. E3.26e. Span AB has both positive and negative components; span BC has a negative component; and span CD has a positive component. These segments are plotted in Fig. E3.26e. 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 269 Fig. E3.26e Moment envelope Example 3.27 Cantilever Construction—Concentrated Loading Given: The three-span symmetrical scheme shown in Fig. E3.27a. There are two moment releases located symmetrically with respect to the center of the center span. This structure is statically determinate: Member cd functions as a simply supported member; segments bc and de act as cantilevers in providing support for member cd. The structure is used for spanning distances which are too large for a single span or a combination of two spans. Determine: A method for selecting L1 and the location of the moment releases corresponding to a concentrated live loading P for a given length, given LT. Solution: The optimal geometric arrangement is determined by equating the maximum moments in the exterior and interior spans. Given the total crossing length, LT, one generates a conceptual design by selecting L1 and α which defines the location of the hinges. The remaining steps are straightforward. One applies the design live loading, determines the moment diagrams, and designs the corresponding cross-sections. The local topography may control where the interior supports may be located. Also, we assume here that we are not concerned in choosing L1 and describe below how one can utilize moment diagrams to arrive at an optimal choice for L1 and α. We consider the design load to be a single concentrated force that can act on any span. The approach that we follow is to move the load across the total span and determine the moment diagram for each load position. 270 3 Statically Determinate Beams Fig. E3.27a When the load is on ab, member ab functions as a simply supported beam, and we know from the previous example that the critical location is at mid-span. As the load moves from b to c, bc acts like a cantilever, and the critical location is point c. Lastly, applying the load at the midpoint of c, d produces the maximum moment for cd. Since the structure is symmetrical, we need to move the load over only one-half the span. Moment diagrams for these cases are shown in Figs. E3.327b, E3.327c, and E3.327d. Fig. E3.27b Moment diagram—load on member AB 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 271 Fig. E3.27c Moment diagram—load on member BC Fig. E3.27d Moment diagram—load on member CD Based on these analyses, the peak moments for the individual spans are PL1 Mjbc ¼ ... PL1 Mjab ¼ 4 ... 272 3 Statically Determinate Beams ... The "optimal" center span is 2L1 þ L2 ¼ LT ... ¼0:429LT 7 If the moment releases can be located such that these span lengths can be realized, the design is optimal for this particular design loading. We want to emphasize here that we are using an analytical approach to generate design scenarios. This approach is useful for gaining insight about the behavior, which provides the basis for rational design. One could have solved this problem by iterating through various geometries, i.e., assuming values for α and L1, but the strategy described above is a better structural engineering approach. Example 3.28 Cantilever Construction—Uniform Design Loading Given: The three-span symmetrical structure shown in Fig. E3.28a. Fig. E3.28a Determine: The optimal values of L1 and α corresponding to a uniform live loading. Solution: Using the results of the previous example, first, we establish the influence lines for the moment at mid-span of ab (M1-1), at point b (Mb), and at mid-span of cd (M2-2). These are plotted in Fig. E3.28b. 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 273 Fig. E3.28b Influence lines We suppose that the uniformly distributed live load can be applied on an arbitrary segment of a span. We start with the side span, ab. Based on the influence line, we need to load span ab (Fig. E3.28c). Next, we load the center span ... for Mb and Mcd (Fig. E3.28d). The third option is to load the center span (Fig. E3.28e). The peak values for these loading schemes are ... 274 3 Statically Determinate Beams Fig. E3.28c Moment diagram ... remaining steps are the same as for the previous example. We want to use a constant crosssection for the total span and therefore equate the design moments. This step results in values for α and L1. 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 275 Setting Mbc ¼ Mcd results in ... Setting Mab ¼ Mcd leads to ... These results are close to the values based on a single concentrated load. Examples 3.27 and 3.28 illustrate an extremely important feature of statically determinate structures. The reactions and internal forces produced by a specific loading depend only on the geometric parameters of the components that comprise the structure. They are independent of the properties of the components of the structure. This allows one to obtain a more favorable internal force distribution by adjusting the geometry as we did in the examples. In Part II of the text, we rework these problems using beams which have constant cross-sections. 3.10.2.1 Multiple Concentrated Loads We consider next the case where there are two concentrated forces. This loading can simulate the loading due to a two axle vehicle. The notation is defined in Fig. 3.51. Fig. 3.51 Two concentrated forces 276 3 Statically Determinate Beams The resultant is R ¼ P1 + P2 located e units from the line of action of P2. where e is found using: ... The moment diagram for a set of concentrated loads is piecewise linear with peak values at the points of application of the forces. One can compute M1 and M2 for a range of x values and determine the values of x corresponding to a maximum value of a particular moment by differentiating the corresponding moment expression with respect to x and setting the derivative equal to zero. Fig. 3.52 Moment diagram—arbitrary position of loading 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 277 These expressions can be written ... ð3:60Þ Maximum value of ... We can interpret the critical location for the maximum value of M2 from the sketch shown in Fig. 3.53a. The force P2 is located e/2 units to the left of mid-span and the line of action of the resultant is e/2 units to the right of mid-span. A similar result applies for M1. P1 is positioned such that P1 and R are equidistant from mid-span as shown in Fig. 3.53b. The absolute maximum live load moment is found by evaluating M1 and M2 and using the largest value. In most cases, the absolute maximum moment occurs at the point of application of the largest force positioned according to (3.60) and (3.61). Example 3.29 Illustration of Computation of Maximum Moments for Two-Force Loading Given: The beam shown in Fig. E3.29a and the following data. ... Fig. E3.29a Determine: The absolute maximum moment. Solution: The resultant is located ... 278 3 Statically Determinate Beams Fig. 3.53 Critical location of loading for maximum bending moments. ... 279 280 3 Statically Determinate Beams The critical loading position for M1 is shown in Fig. E3.29c. Fig. E3.29c It follows that the absolute maximum moment occurs when P2 is positioned 18.6 ft from the left support. This point is close to mid-span (Fig. E3.29d). Fig. E3.29d The analysis for the case of three concentrated loads proceeds in a similar way. Figure 3.54 shows the notation for three concentrated loads. We generate expressions for the moments at points ①, ②, and ③ for an arbitrary position of the loading defined by x and then determine the locations of maximum moments by differentiating these expressions. First, we locate the resultant. ... 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 281 Fig. 3.54 Notation and moment diagram—three concentrated loads The moments at locations 1, 2, and 3 are functions of x. ... Differentiating each expression with respect to x and setting the result to zero leads to the equations for the critical values of x that correspond to relative maximum values of the moments. The positions of the loading corresponding to these three values of x are plotted in Fig. 3.55. Note that the results are similar to the two concentrated load case. Once the value of x is known, one can evaluate (3.63) for each value of x in order to establish the absolute maximum moment. 282 Fig. 3.55 Possible locations of loading for maximum moments. 3 Statically Determinate Beams 3.10 Influence Lines and Force Envelopes for Statically Determinate Beams 283 Example 3.30 Given: The beam shown in Fig. E3.30a. Fig. E3.30a Determine: The absolute maximum moment in the beam caused by 1. A truck moving across the span (Fig. E3.30b). 2. A uniformly distributed dead load of wD ¼ 2.4 kip/ft in addition to the truck loading. Solution: Part (1): The critical truck loading position is shown in Fig. E3.30c. The corresponding bending moment diagram is plotted below. The absolute maximum moment occurs 2.3 ft from the center of the span. Mmax ¼ 806.7 kip ft. Part (2): The maximum moment diagram for uniform loading is parabolic, with a maximum value at mid-span. ... We estimate the peak moment by adding corresponding moment values from Figs. E3.30c and E3.30d. 284 3 Statically Determinate Beams ... Fig. E3.30c Moment diagram for truck load (d) Moment diagram for dead load (e) Combined moment diagram When there are multiple loadings, it is more convenient to generate discrete moment envelope using a computer-based analysis system. The discrete moment envelope for the combined (dead + truck) loading is plotted below (Fig. E3.30e). Scanning the envelope shows that the maximum moment occurs at x ¼ 30 ft and is 1882.6 kip ft. This result shows that it was reasonable to superimpose the moment diagrams for the two loadings. Fig. E3.30e Discrete moment envelope for the combined (dead + truck) loading 3.11 Summary 3.11.1 Objectives of the Chapter • To develop analytical and computational methods for the analysis of beams subjected to transverse loading. Of particular interest are the internal forces (shear, bending, and twisting moments), and the displacements. • To introduce the concepts of influence line and force envelope which are needed to establish design values for beam cross-sections. 3.11.2 Key Facts and Concepts • A stable statically determinate beam requires three nonconcurrent displacement restraints. There are three reaction forces which are determined using the static equilibrium equations. • External loads are resisted by internal forces acting on a cross-section. For planar loading, these quantities consist of an axial force, F, a transverse shear force, V, and a bending moment, M. One can establish the magnitude of these variables using the static equilibrium equations. Alternatively, one can start with the following pair of differential equilibrium equations, dV ¼w dx dM ¼ V dx 286 3 Statically Determinate Beams Integrating between points 1 and 2 leads to V2 V1 ¼ ∫w dx2 M2 M1 ¼ ∫V dx V dx x1 The first equation states that the change in shear is equal to the area under the loading diagram. The second equation states that the change in moment is equal to the area under the shear diagram. • Planar bending results in a transverse deflection, v, and rotation, θ (x). When the beam is slender, these variables are related by dθ/dx ¼ M/EI where EI is the second moment of area for the section. Given M(x), one determines v(x) by integrating this expression and using the two boundary conditions on v. • The transverse displacement at a particular point can be determined using the Principle of Virtual Forces specialized for beams. δ M dP ¼ ∫ δM dx EI Here, δ is the desired displacement, δP is a virtual force in the direction of d, and δM is the virtual moment corresponding to δP. One usually employs numerical integration when the integral is complex. • An influence line is a plot of the magnitude of a particular internal force quantity, say the bending moment at a specific location, vs. the position of a unit concentrated load as it moves across the span. It is useful for establishing the peak magnitude of the force quantity at that location. • A force envelope is a plot of the maximum positive and negative values of a particular force quantity, say the bending moment, at different sections along the beam. This is used to determine crosssectional properties. 3.12 Problems For the beams defined in Problems 3.1–3.20, compute the reactions and draw the shear and moment diagrams. Problem 3.1 3.12 Problems Problem 3.2 Problem 3.3 Problem 3.4 287 288 Problem 3.5 Problem 3.6 Problem 3.7 Problem 3.8 3 Statically Determinate Beams 3.12 Problems Problem 3.9 Problem 3.10 Problem 3.11 Problem 3.12 289 290 Problem 3.13 Problem 3.14 Problem 3.15 Problem 3.16 3 Statically Determinate Beams 3.12 Problems Problem 3.17 Problem 3.18 Problem 3.19 Problem 3.20 Member BD is rigidly attached to the beam at B. 291 292 3 Statically Determinate Beams Problem 3.21 Determine the maximum bending moment. Assume EI is constant. Problem 3.22 Determine the maximum bending moment. Do the results depend on either E or I? Justify your response. For the beams defined in Problems 3.23–3.26, use the Table 3.1 to determine the deflection and rotation measures indicated. Assume EI is constant. 3.12 Problems Problem 3.23 θB , vB I ¼ 200 in.4 , E ¼ 29, 000 ksi Problem 3.24 Problem 3.25 Problem 3.26 θB , vB I ¼ 200 in.4 , E ¼ 29, 000 ksi θA , vC I ¼ 80 106 mm4 , θD , vC I ¼ 80 106 mm4 , E ¼ 200 GPa E ¼ 29, 000 ksi For the beams defined in Problems 3.27–3.35, use the virtual force method to determine the deflection and rotation measures indicated. 294 3 Statically Determinate Beams Problem 3.27 θD , vC I ¼ 80 106 mm4 , E ¼ 200 GPa Problem 3.28 θB , vB I ¼ 120 in.4 , E ¼ 29, 000 ksi Problem 3.29 θA , vC I ¼ 80 106 mm4 , E ¼ 200 GPa Problem 3.30 Problem 3.31 Problem 3.32 295 θC , vD I ¼ 200 in.4 , E ¼ 29, 000 ksi θD , vB I ¼ 100 106 mm4 , E ¼ 200 GPa θB , vC I ¼ 100 106 mm4 , E ¼ 200 GPa 296 Problem 3.33 Problem 3.34 Problem 3.35 3 θB , vE I ¼ 300 in.4 , E ¼ 29, 000 ksi θC , θE , and vE I ¼ 160 106 mm4 , θB , vB I ¼ 120 in.4 , E ¼ 29, 000 ksi Statically Determinate Beams E ¼ 200 GPa Use computer software to determine the deflected shape for the beams defined in Problems 3.36–3.39. Assume EI is constant. 3.12 Problems Problem 3.36 Problem 3.37 Problem 3.38 Problem 3.39 297 298 3 Statically Determinate Beams Problem 3.40 Determine the value of I require to limit the vertical deflection at B to 1/2 in. E ¼ 29, 000 ksi. Problem 3.41 (a) Solve Problem 3.39 using computer software. Consider different sets of values for EI. Show that the magnitude of the deflection varies as 1/EI. Assume P ¼ 100 kN, and L ¼ 8 m. (b) Suppose the peak deflection is specified. How would you determine the appropriate value of I? Problem 3.42 Utilize symmetry to sketch the deflected shape of the beam. EI is constant. 3.12 Problems 299 Problem 3.43 Determine the value of I required to limit the vertical deflection at the midspan of AB to 20 mm. E ¼ 200 GPa and I ¼ 160(10)6 mm4. Sketch the deflected shape. Assume EI is constant. Problem 3.44 Determine the vertical deflection at midspan. Assume EI is constant. Problem 3.45 Problem 3.46 Use the Principle of Virtual Forces and evaluate the corresponding integral with the trapezoidal rule. mx Take w0 ¼ 10 kip=ft, L ¼ 20 ft, I0 ¼ 1000 in:4 , E ¼ 29, 000 ksi, I ¼ I0 1 þ cos : 2L 300 3 Statically Determinate Beams Problem 3.47 Assume AB is a "deep" beam. I and A are constant. Determine the analytical solution for β (the rotation of the cross-section) and v. Problem 3.48 1. Determine βt (the rotation of the cross-section about the longitudinal axis) at B due to the concentrated torque at C. Mx. 2. Suppose a distributed torsion, mt, is applied along A–B. Determine Mt(x). Take mt ¼ sin 2L 3. Determine βt at B due to the distributed torsional loading. Problem 3.49 Draw the influence lines for: (a) Reaction at A (b) Moment at B (c) Shear at D 3.12 Problems 301 Problem 3.50 Draw the influence lines for the moment at E and shear at E. Problem 3.51 For the beams shown, determine the moment envelope corresponding to a single concentrated load moving across the span. Problem 3.52 (a) Draw the influence lines for moment at F and moment at B. (b) Draw the moment envelope. Suppose a uniformly distributed dead load of 18 kN/m and uniformly distributed live load of 30 kN/m are placed on the beam. Use above results to determine the maximum positive and negative moment at point F and point B. Also show the position of the live load on the beam for these limiting cases. Problem 3.53 Suppose a uniformly distributed live load of 1.2 kip/ft and uniformly distributed dead load of 0.8 kip/ft are placed on the beam. Use the influence line to determine the loading pattern that results in the maximum and minimum moment at E. 302 3 Statically Determinate Beams Problem 3.54 For the beams shown, determine the moment envelope corresponding to a single concentrated load moving across the span. Problem 3.55 Determine the maximum possible moment in the 40 ft span beam as the loading system shown moves across the span. Use either the analytical approach or the moment envelope. Problem 3.56 Determine the maximum possible moment in the 20 m span beam as the loading system shown moves across the span. 3.12 Problems 303 Problem 3.57 Determine the maximum possible moment in a 80 ft span beam as the loading system shown moves across the span. Assume a uniform dead load of 2 kip/ft also acts on the span. Problem 3.58 For the beam shown below: (a) Draw the influence line for: (i) The vertical reaction at support D, and (ii) The moment at B. Suppose a uniformly distributed dead load of 10 kN/m and uniformly distributed live load of 20 kN/m are placed on the beam. Use the results to determine the maximum values of the reaction at D, and the maximum positive and negative moments at B. Also show the position of the live load on the beam. (c) Establish the moment envelope corresponding to a single concentrated load. Problem 3.59 For the beam shown below Determine the influence line for: (a) The vertical reaction at C (b) The moment at D 304 3 Statically Determinate Beams If a uniformly distributed live load of 1.8 kip/ft and uniformly distributed dead load of 1.4 kip/ft are placed on the beam, use the above results to determine the maximum and minimum values of (a) The vertical reaction at C (b) The moment at D Problem 3.60 Using computer software, determine the influence line for the vertical reaction at C and the moment at B. Take the stiffness of the spring as k ¼ 5 m. Assume EI is constant. Hint: Apply a unit load at x ¼ 5 m and determine the deflected shape. This is a scaled version of the influence line. Verify by moving the load and recomputing the displacement at x ¼ 5 m. References 1. Tauchert TR. Energy principles in structural mechanics. New York: McGraw-Hill; 1974. 2. Mathab 7.13, Engineering calculations software. 4 Statically Determinate Plane Frames Abstract Plane frame structures are composed of structural members which lie in a single plane. When loaded in this plane, they are subjected to both bending and axial action. In this chapter we discuss how one establishes the internal forces and displacements of statically determinate frames under gravity and lateral loadings. We describe in this chapter analysis strategies which are useful for analyzing statically determinate single-story frames. Numerous examples illustrating the response are presented to provide the reader with insight as to the behavior of these structural types. We also describe how the Method of Virtual Forces can be applied to compute displacements of frames. The theory for frame structures is based on the theory of beams presented in Chap. 3. Later in Chaps. 9, 10, and 15, we extend the discussion to deal with statically indeterminate frames and space frames. 4.1 Definition of Plane Frames The two dominant planar structural systems are plane trusses and plane frames. Plane trusses were discussed in detail in Chap. 2. Both structural systems are composed of members which lie in a single plane. The systems differ in the way the individual members are connected and loaded. Trusses are loaded only at their nodes (joints) for truss structures. Consequently, the members forces are purely axial. Frame structures behave in a completely different way. The loading is applied directly to the members, resulting in internal shear and moment as well as axial forces in the members. Also, a set of members may experience predominately bending action; they are called "beams." Another set may experience predominately axial action; they are called "columns." The typical building frame is composed of a combination of beams and columns. Frames are categorized partly by their geometry and partly by the nature of the member/member connection, i.e., pinned vs. rigid connection. Figure 4.1 illustrates some typical rigid plane frames used mainly for light manufacturing facilities, warehouses, and office buildings. We generate threedimensional frames by suitably combining plane frames. # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_4 305 306 4 Statically Determinate Plane Frames Fig. 4.1 Typical plane building frames. (a) Rigid portal frame. (b) Rigid multi-bay portal frame. (c) Multistory rigid frame. (d) Multistory braced frame Fig. 4.2 Frames A-frame Figure 4.2 shows an A-frame, named obviously for its geometry. This frame has three members ab, bc, and de that are pinned together at points d, b, and e. Loads may be applied at the connection points, such as b, or on a member, such as d. A-frames are typically used at the base of their legs, such as at a and c. Because of the nature of the loading and restraints, the members in an A-frame generally experience bending as well as axial forces. We discuss this structural type later in Example 4.13. Pitched roof frames are used where it is necessary to provide a large open clearance in the interior of the portal frame, and also to improve the aesthetics, a more open space is created by pitching the top members as shown in Fig. 4.3. Pitched roof frames are used in buildings for churches, gymnasia, and exhibition halls. Architects tend to prefer them for aesthetic reasons. 4.2 Statical Determinacy: Planar Loading All the plane frames that we have discussed so far can be regarded as rigid bodies in the sense that if they are adequately supported, the only motion they experience when a planar load is applied will 4.2 Statical Determinacy: Planar Loading 307 Fig. 4.3 Gable (pitched roof) frames Fig. 4.4 Statically determinate support schemes Fig. 4.5 Statically indeterminate support schemes—plane frames be due to deformation of the members. Therefore, we can support them with only three nonconcurrent displacement restraints. One can use a single, fully fixed support scheme, or a combination of hinge and roller supports. Examples of "adequate" support schemes are shown in Fig. 4.4. All the schemes are statically determinate. In this case, one first determines the reactions and then analyzes the individual members. If more than three displacement restraints are used, the plane frame is statically indeterminate. In many cases, two hinge supports are used for portal frames and gable frames. These structures are statically indeterminate to the first degree. Examples of these support schemes are listed in Fig. 4.5. One can reduce the structure to statically determinate structures by inserting a hinge which acts as a moment release. We refer to these modified structures as 3-hinge frames (see Fig. 4.6). Statical determinacy is evaluated by comparing the number of unknown forces with the number of equilibrium equations available. For a planar member subjected to planar loading, there are three internal forces: axial, shear, and moment. Once these force quantities are known at a point, the force quantities at any other point in the member can be determined using the equilibrium equations. Figure 4.7 illustrates the use of equilibrium equations for the member shown. It follows that there are three force unknowns for each member of a planar frame subjected to planar loading. We define a node (joint) as the intersection of two or more members, or the end of a member connected to a support. A node is acted upon by the member forces and the external loads and reactions. Figure 4.8 illustrates the forces acting on node B. 308 4 Statically Determinate Plane Frames Fig. 4.6 3-hinge frames Fig. 4.7 Free body diagram—member forces Fig. 4.8 Free body diagram—node B These nodal forces comprise a general planar force system for which there are three equilibrium equations available; summation of forces in two nonparallel directions and summation of moments. Summing up force unknowns, we have three for each member plus the number of displacement restraints. Summing up equations, there are three for each node plus the number of force releases (e.g., moment releases) introduced. Letting m denote the number of members, r the number of displacement restraints, j the number of nodes, and n the number of releases, the criterion for statical determinacy of rigid plane frames can be expressed as 3m þ r ¼ 3j ð4:1Þ We apply this criterion to the portal frames shown in Figs. 4.4a, 4.5a, and 4.6b. For the portal frames in Fig. 4.4a m ¼ 3, r ¼ 3, j¼4 4.2 Statical Determinacy: Planar Loading 309 Fig. 4.9 Indeterminate portal and A-frames Fig. 4.10 Statically determinate portal and A-frame and the criterion is satisfied. For the corresponding frame in Fig. 4.5a, r ¼ 4, m ¼ 3, j ¼ 4 Therefore, this structure is indeterminate to the first degree. The 3-hinge frame shown in Fig. 4.6b has n ¼ 1. Inserting the moment release reduces the number of unknowns and now the resulting structure is statically determinate. For the frame shown in Fig. 4.9a, m ¼ 4, r ¼ 4, j ¼ 5 The structure is therefore indeterminate to the third degree. m ¼ 3, r ¼ 6, j¼4 Fig. 4.9b is indeterminate to the third degree. m ¼ 4, r ¼ 4, j¼5 n¼1 Equation (4.1) applies to rigid plane frames, i.e., where the members are rigidly connected to each other at all nodes. The members of an A-frame are connected with pins that allow relative rotation and therefore A-frames are not rigid frames. We establish a criterion for A-frame type structures following the same approach described above. Consider the frame shown in Fig. 4.2. It consists of three rigid members which are connected with pins. Each pin introduces two force unknowns. The number of force unknowns is equal to 2np þ r, where np is the number of pins. Since there are three equations for each member, the total number of equations is equal to 3m. It follows that 2np þ r ¼ 3m ð4:2Þ 310 4 Statically Determinate Plane Frames for statical determinacy of A-frame type structures. Applying this criterion to the structure shown in Fig. 4.2, one has np ¼ 3, r ¼ 3, m ¼ 3, and the structure is statically determinate. The A-frame in Fig. 4.9b has one extra support, r ¼ 4, and therefore it is statically indeterminate to the first degree. If we add another member at the base, as shown in Fig. 4.9c, np ¼ 5, r ¼ 3, m ¼ 4, and the structure is indeterminate to the first degree. 4.3 Analysis of Statically Determinate Frames In this section, we illustrate with numerous examples the analysis process for statically determinate frames such as shown in Fig. 4.10a. In these examples, our primary focus is on the generation of the internal force distributions. Of particular interest are the location and magnitude of the peak values of moment, shear, and axial force since these quantities are needed for the design of the member cross sections. We find the reactions by enforcing the global equilibrium equations. Once the reactions are known, we determine the end actions for each member and then compute the moment and shear distributions for the members. The sign conventions for bending moment, transverse shear, and axial force are defined in Fig. 4.10b. For the following examples illustrate this analysis strategy. Later, we present analytical solutions which are useful for developing an understanding of the behavior. Fig. 4.10 (a) Typical frame. (b) Sign convention for the bending moment, transverse shear, and axial force 4.3 Analysis of Statically Determinate Frames 311 Example 4.1 UnsymmetricalCantilever Frame Given: The structure defined in Fig. E4.1a. Fig. E4.1a Determine: The reactions and draw the shear and moment diagrams. Solution: We first determine the reactions at A, and then the shear and moment at B. These results are shown in Figs. E4.1b and E4.1c. Once these values are known, we can determine the moment diagram for member BC. Using these results, the moment diagram for member AB can be obtained. 4 Statically Determinate Plane Frames 312 Fig. E4.1b Reactions Fig. E4.1c End actions Fig. E4.1d Shear and moment diagrams Example 4.2 Symmetrical Cantilever Frame Given: The structure defined in Fig. E4.2a. Fig. E4.2a 4 Statically Determinate Plane Frames 4.3 Analysis of Statically Determinate Frames 313 Determine: The reactions and draw the shear and moment diagrams. Solution: We determine the reactions at A and then the end moments at B. The results are shown in Figs. E4.2b and E4.2c. X F ¼ 0 RAx ¼ 0 X F ¼ 0 RAy ¼ 0 X MA ¼ 0 MA ¼ 0 Fig. E4.2b Reactions Fig. E4.2c End actions 314 4 Statically Determinate Plane Frames Finally, the shear and moment diagrams for the structure are plotted in Fig. E4.2d. Note that member AB now has no moment, just an axial compressive force of 60 kN. Fig. E4.2d Shear and moment diagrams 4.3 Analysis of Statically Determinate Frames 315 Example 4.3 Angle-Type Frame Segment Given: The frame defined in Fig. E4.3a. Fig. E4.3a Determine: The reactions and draw the shear and moment diagrams. Solution: We determine the vertical reaction at C by summing moments about A. The reactions at A follow from force equilibrium considerations (Fig. E4.3b). 4.3 Analysis of Statically Determinate Frames X MA ¼ 0 ... RC ¼ 23 kip ↑ RAy ¼ 17 kip ↑ Fig. E4.3b Reactions Next, we determine the end actions and then the axial, shear, and moment diagrams. ... Fig. E4.3c End actions Lastly, we determine the location of the maximum moment in segment AB by noting that the shear is zero. 316 4 Statically Determinate Plane Frames 23 ... ¼ 0 ! x1 ¼ 11:5 ft Then, Mmax ¼ 23ð11:5Þ 2ð11:5Þ2 ¼ 132:25 kip ft 2 Fig. E4.3d Shear and moment diagrams Example 4.4 Simply Supported Portal Frame Given: The portal frame defined in Fig. E4.4a. Determine: The shear and moment distributions. Fig. E4.4a Solution: The reaction at D is found by summing moments about A. We then determine the reactions at A using force equilibrium considerations. Figure E4.4b shows the result. 4.3 Analysis of Statically Determinate Frames X X X MA ¼ 0 RD ð32Þ þ ... RD ¼ 17:25 ... RAy ¼ 14:75 " Fig. E4.4b Reactions Isolating the individual members and enforcing equilibrium leads to the end forces and moments shown in Fig. E4.4c. Fig. E4.4c End actions We locate the maximum moment in member BC. Suppose the moment is a maximum at x ¼ x1. Setting the shear at this point equal to zero leads to 318 4 Statically Determinate Plane Frames 17:25 x1 ð1Þ ¼ 0 ! x1 ¼ 17:25 ft ð1Þð17:25Þ2 ¼ 148:78 kip ft 2 The shear and moment diagrams are plotted in Fig. E4.4d. Then, Mmax ¼ 17:25ð17:25Þ Fig. E4.4d Shear and moment diagrams Example 4.5 A 3-Hinge Portal Frame Given: The 3-hinge frame defined in Fig. E4.5a. Determine: The shear, moment, and axial force diagrams. Fig. E4.5a Solution: Results for the various analysis steps are listed in Figs. E4.5b, E4.5c, E4.5d, E4.5e, E4.5f, and E4.5g. Step 1: Reactions at D and A The vertical reaction at D is found by summing moments about A. 4.3 Analysis of Statically Determinate Frames X MA ¼ 0 RDy ð32Þ ð1Þð32Þð16Þ 2ð20Þ ¼ 0 319 RDy ¼ 17:25 kip " Fig. E4.5b Next, we work with the free body diagram of segment ECD. Applying the equilibrium conditions to this segment results in X ME ¼ 0 ð17:25Þð16Þ ð1Þð16Þð8Þ RDX ð20Þ ¼ 0 RDX ¼ 7:4 kip ← X Fx ¼ 0 FE ¼ RDX ¼ 7:4 kip ! X Fy ¼ 0 V E þ 17:25 ð1Þð16Þ ¼ 0 V E ¼ 1:25 kip # Fig. E4.5c With the internal forces at E known, we can now proceed with the analysis of segment ABE. 320 4 Statically Determinate Plane Frames X F ¼ 0 RAx þ 2 7:4 ¼ 0 RAx ¼ 5:4 kip ! X Fy ¼ 0 RAy 1:25 ð1Þð16Þ ¼ 0 RAy ¼ 14:75 kip " Fig. E4.5d Reactions at A Steps 2–3: End actions at B and C Fig. E4.5e Steps 3: Shear and moment diagrams 4.3 Analysis of Statically Determinate Frames 321 BC. 14:75 1014:75Þ2 ¼ 108 ¼ 0 Þ x1 ¼ 14:75 ft ð1Þð14:75Þ2 The corresponding shear and moment diagrams are listed in Fig. E4.5g. Then, Mmax ¼ 14:75ð14:75Þ Fig. E4.5f Fig. E4.5g Shear and moment diagrams Example 4.6 Frame with Overhang Given: The frame shown in Fig. E4.6a. Fig. E4.6a Solution: Results for the various analysis steps are listed in Figs. E4.6b, E4.6c, and E4.6d. 321 322 4 Statically Determinate Plane Frames X X X MA ¼ 0 RD ð10Þ 80ð6Þ ð15Þð13Þð6:5Þ ¼ 0 Fx ¼ 0 RAx ¼ 0 kN ← X Fy ¼ 0 RAy þ 131:55 ð15Þð13Þ ¼ 0 Statically Determinate Frames RD ¼ 131:55 kN " RAy ¼ 63:45 kN " Fig. E4.6b Reactions Fig. E4.6c End actions First, we locate the maximum moment in member BC. 63:45 ð15Þx1 ¼ 0 ! x1 ¼ 4:23 m Then, Mmax ¼ 63:45ð4:23Þ ð15Þð4:23Þ2 ¼ 48 ¼ 182 kN m 2 4.3 Analysis of Statically Determinate Frames 323 The corresponding shear and moment diagrams are listed in Fig. E4.6d. Fig. E4.6d Shear and moment diagrams 4.3.1 Behavior of Portal Frames: Analytical Solution The previous examples illustrated numerical aspects of the analysis process for a single-story statically determinate portal frames. For future reference, we list below the corresponding analytical solutions (Figs. 4.11, 4.12, 4.13, and 4.14). We consider both gravity and lateral loading. These solutions are useful for reasoning about the behavior of this type of frame when the geometric parameters are varied. Portal frame—Gravity loading: Shown in Fig. 4.11 Portal frame—lateral loading: Shown in Fig. 4.12 3-hinge portal frame—lateral loading: Shown in Fig. 4.13 3-hinge portal frame—gravity loading: (a) Geometry and loading. (b) Reactions. (c) Shear diagram. (d) Moment diagram 324 Fig. 4.12 Statically determinate portal frame under lateral loading. (a) Geometry and loading. (b) Reactions. (c) Shear diagram. (d) Moment diagram Fig. 4.13 Statically determinate 3-hinge frame under gravity loading. (a) Geometry and loading. (b) Reactions. (c) Shear diagram. (d) Moment diagram 4 Statically Determinate Plane Frames 4.3 Analysis of Statically Determinate Frames 325 Fig. 4.14 Statically determinate 3-hinge frame under lateral loading. (a) Geometry and loading. (b) Reactions. (c) Shear diagram. (d) Moment diagram Fig. 4.15 Variable crosssection 3-hinge frame These results show that the magnitude of the peak moment due to the uniform gravity load is the same for both structures but of opposite sense (Figs. 4.11 and 4.13). The peak moment occurs at the corner points for the 3-hinge frame and at mid-span for the simply supported frame which behaves as a simply supported beam. The response under lateral loading is quite different (Figs. 4.12 and 4.14). There is a 50 % reduction in peak moment for the 3-hinge case due to the inclusion of an additional horizontal restraint at support D. For the 3-hinge frame, we note that the bending moment due to gravity loading is symmetrical. In general, a symmetrical structure responds symmetrically when the loading is symmetrical. We also note that the bending moment diagram for lateral loading applied to the 3-hinge frame is anti-symmetrical. Both loadings produce moment distributions having peaks at the corner points. In strength-based design, the cross-sectional dimensions depend on the design moment; the deepest section is required by the peak moment. Applying this design approach to the 3-hinge frame, we can use variable depth members with the depth increased at the corner points and decreased at the supports and mid-span. Figure 4.15 illustrates a typical geometry. Variable depth 3-hinge frames are quite popular. We point out again here that the internal force distribution in statically determinate structures depends only on the loading and is independent of the cross-sectional properties of the members. Therefore, provided we keep the same geometry (centerline dimensions), we can vary the crosssection properties for a 3-hinge frame without changing the moment distributions. 326 4.4 4 Statically Determinate Plane Frames Pitched Roof Frames In this section, we deal with a different type of portal frame structure: the roof members are sloped upward to create a pitched roof. This design creates a more open interior space and avoids the problem of rain water pounding or snow accumulating on flat roofs. Figure 4.16 shows the structures under consideration. The first structure is a rigid frame with a combination of pin and roller supports; the second structure is a 3-hinge frame. Both structures are analyzed by first finding the reactions and then isolating individual members to determine the end forces, and the internal force distributions. Fig. 4.16 Pitched roof frames 4.4.1 Member Loads Typical loadings that may be applied to an inclined member are illustrated in Fig. 4.17. They act either in the vertical direction or normal to the member. In the vertical direction, they may be defined either in terms of the horizontal projection of the length of the member or in terms of the member arc length. 4.4 Pitched Roof Frames Fig. 4.17 Loading on an inclined member. (a) Vertical load per horizontal projection. (b) Vertical load per length. (c) Normal load per length 327 328 4 Statically Determinate Plane Frames When computing the reactions, it is convenient to work with loads referred to horizontal and vertical directions and expressed in terms of the horizontal projection. The first form (a) is already in this form. For the w2 load, we note that ds ¼ dx cos θ Then, w2 ds ¼ w2 cos θ dx Lastly, the w3 load is normal to the member. We project it onto the vertical and horizontal directions and then substitute for ds. w3 ds cos θ ¼ w3 dx w3 ds sin θ ¼ w3 tan θ dx ¼ w3 dx tan θ ¼ w3 dy Fig. 4.18 shows the equivalent loads. 4.4 Pitched Roof Frames Fig. 4.18 Equivalent vertical member loadings. (a) Per horizontal projection. (b) Per length. (c) Normal load per length When computing the axial force, shear, and moment distributions in a member, it is more convenient to work with loads referred to the normal and tangential directions and expressed in terms of the member arc length. The approach is similar to the strategy followed above. The results, as summarized, below are (Fig. 4.19): Vertical—per horizontal projection length: w1, n ¼ w1 cos θ2 w1, t ¼ w1 cos θ sin θ Member loading: w2, n ¼ w2 cos θ w2 t ¼ w2 sin θ w3, n ¼ w3 w3, t ¼ 0 330 4 Statically Determinate Plane Frames Fig. 4.19 Equivalent normal and tangential member loadings. (a) Vertical per projected length. (b) Vertical per length. (c) Normal 4.4.2 Analysis of Pitched Roof Frames Analytical solutions for the gable frame are listed below. Gravity loading: The moment distribution for uniform gravity loading on the rigid frame is given. Results are listed in Figs. 4.20 and 4.21. Lateral Loading: Results are shown in Figs. 4.22 and 4.23. Fig. 4.20 Simply supported gable frame. (a) Structure and loading. (b) Moment diagram Fig. 4.21 Simply supported rigid frame under gravity loading. (a) Structure and loading. (b) Moment diagram 332 Fig. 4.22 Simply supported rigid frame— lateral loading. (a) Structure and loading. (b) Moment diagram Fig. 4.23 3-Hinge frame—lateral loading. (a) Structure and loading. (b) Moment diagram 4 Statically Determinate Plane Frames 4.4 Pitched Roof Frames Example 4.7 333 Simply Supported Gable Frame—Lateral Load Given: The gable frame with the lateral load defined in Fig. E4.7a. Determine: The shear, moment, and axial force diagrams. Fig. E4.7a Solution: Moment summation about A leads to the vertical reaction at E. The reactions at A follow from force equilibrium considerations. Next, we determine the end forces and moments for the individual members. Lastly, we generate the shear and moment diagrams. Results for the various steps are listed in Figs. E4.7b, E4.7c, E4.7d, and E4.7e. Step 1: Reactions Fig. E4.7b Reactions 334 Step 2: End forces—global frame Fig. E4.7c End forces—global frame Step 3: Member forces—member frame Fig. E4.7d End forces in local member frame 4 Statically Determinate Plane Frames 4.4 Pitched Roof Frames Step 4: Internal force distributions Fig. E4.7e Force distributions Example 4.8 A 3-Hinge Gable Frame—Lateral Loading Given: The 3-hinge gable frame defined in Fig. E4.8a. Determine: The shear, moment, and axial force diagrams. Fig. E4.8a 335 336 4 Statically Determinate Plane Frames Solution: Step 1: Reactions The reactions (Fig. E4.8b) are determined by summing moments about A and C and applying the force equilibrium conditions. Fig. E4.8b Reactions Step 2: End forces—global frame (Fig. E4.8c) Fig. E4.8c End forces Step 3: End forces—local member frame 4.4 Pitched Roof Frames 337 Fig. E4.8d shows the end forces and moments resolved into components referred to the local member frame. Fig. E4.8d End actions in local member frame Step 4: Internal force distributions Fig. E4.8e Force distributions Note that the 3-hinge gable frame carries a lower value of peak moment. Example 4.9 Simply Supported Gable Frame—Unsymmetrical Loading Given: The frame defined in Fig. E4.9a. Determine: The member force distributions. 338 4 Statically Determinate Plane Frames Determine: The member force distributions. Fig. E4.9a Solution: The loading consists of a vertical load per horizontal projection applied to member BC. We determine the reactions at E and A by summing moments about A and E and by enforcing vertical equilibrium. Figure E4.9b shows the end forces and moments. Fig. E4.9b Reactions Next, we determine the end forces and moments for the individual members. Then, we need to resolve the loading and the end forces for members BC and CD into normal and tangential components. The transformed components are shown in Figs. E4.9c and E4.9d. Fig. E4.9c End actions—global frame 4.4 Pitched Roof Frames 339 Fig. E4.9d End actions—local frame The maximum moment in member BC occurs at x1. We determine the location by setting the shear equal to zero. 13:42 0:8x1 ¼ 0) x1 ¼ 16:775 Then, Mmax ¼ 13:42ð16:775Þ 0:8ð16:775Þ2 ð12Þ ¼ 112:56 kip ft Figure E4.9e contains the shear, moment, and axial force diagrams. Fig. E4.9e Internal force diagrams Example 4.10 3-Hinge Gable Frame Given: The 3-hinge gable frame shown in Fig. E4.10a. Determine: The shear and moment diagrams Fig. E4.10a 340 4 Statically Determinate Plane Frames Solution: We analyzed a similar loading condition in Example 4.9. The results for the different analysis phases are listed in Figs. E4.10b, E4.10c, and E4.10d. Fig. E4.10b Reactions Fig. E4.10c End forces 4.5 A-Frames 341 Fig. E4.10d End forces in local frame Fig. E4.10e Shear and moment diagrams for 3-hinge frame. Comparing Fig. E4.10e with Fig. E4.9e shows that there is a substantial reduction in the magnitude of the maximum moment when the 3-hinged gable frame is utilized. 4.5 A-Frames A-frames are obviously named for their geometry. Loads may be applied at the connection points or on the members. A-frames are typically supported at the base of their legs. Because of the nature of the loading and restraints, the members in an A-frame generally experience bending as well as axial force. We consider first the triangular frame shown in Fig. 4.24. The inclined members are subjected to a uniform distributed loading per unit length wg which represents the self-weight of the members and the weight of the roof that is supported by the member. We convert wg to an equivalent uniform horizontal projection w (using (4.3). We start the analysis process by first finding the reactions at A and C. 342 Fig. 4.24 (a) Geometry and loading. (b) A-frame loading and reactions. (c) Free body diagrams. (d) Moment diagram 4 Statically Determinate Plane Frames 4.5 A-Frames 343 Next, we isolate member BC (see Fig. 4.24c). X w L 2 wL L MAt B ¼ þ hFAC ¼ 0 2 2 2 2 FAC ¼ wL 8h The horizontal internal force at B must equilibrate FAC. Lastly, we generate the shear and moment distributions in members AB and BC. Noting Fig. 4.24c, the moment in member BC at location x is given by wL 2h wx2 wL x Fx2 FAC x Mðx Þ¼ ¼ x 2 2 2 The maximum moment occurs at x ¼ L/4 and is equal to Mmax ¼ wL2 32 Replacing w with wg, we express Mmax as Mmax ¼ wL 2 g cos θ 32 As θ increases, the moment increases even though the projected length of the member remains constant. Next, we discuss the frame shown in Fig. 4.25a. There are two loadings: a concentrated force at B and a uniform distributed loading applied to DE. We first determine the reactions and then isolate member DE. Summing moments about A leads to L wL L P wL P ¼ RC L R C ¼ ¼ þ P 2 2 2 2 4 The results are listed below. Noting Fig. 4.25d, we sum moments about B to determine the horizontal component of the force in member DE. L P wL wL L P wL P Fde ¼ ¼ 4 4 2 Fde ¼ PL wL 2 þ 2h 8h The bending moment distribution is plotted in Fig. 4.25e. Note that there is bending in the legs even though P is applied at node A. This is due to the location of member DE. If we move member DE down to the supports A and C, the moment in the legs would vanish. 344 4 Statically Determinate Plane Frames Fig. 4.25 (a) A-frame geometry and loading. (b–d) Free body diagrams. (e) Bending moment distribution 4.6 Deflection of Frames Using the Principle of Virtual Forces The principle of virtual forces specialized for a planar frame structure subjected to planar loading is derived in [1]. The general form is X δ M F d δP ¼ δM þ δF þ δV ds ð4:7Þ EI AE GAs members s 4 Deflection of Frames Using the Principle of Virtual Forces 345 Frames carrying loading primarily by bending action. Axial and shear forces are developed as a result of the bending action, but the contribution to the displacement produced by shear deformation is generally small in comparison to the displacement produced by bending deformation and axial deformation. Therefore, we neglect this term and work with a reduced form of the principle of Virtual Forces. X δ M F d δP ¼ δM þ δF ds ð4:8Þ EI AE members s where δP is either a unit force (displacement) or a unit moment (for rotation) in the direction of the desired displacement d; δM, and δF are the virtual moment and axial force due to δP. The integration is carried out over the length of each member and then summed up. For low-rise frames, where the ratio of height to width is on the order of unity, the axial deformation term is also small in comparison to the bending term. In this case, one neglects the axial deformation term in (4.8) and works with the following form X δ M d δP ¼ δMds ð4:9Þ EI members s Axial deformation is significant for tall buildings, and (4.8) is used for this case. In what follows, we illustrate the application of the Principle of Virtual Forces to some typical low-rise structures. We revisit this topic later in Chap. 9, which deals with statically indeterminate frames. Example 4.11 Computation of Deflection—Cantilever-Type Structure Given: The structure shown in Fig. E4.11a. Assume EI is constant. E ¼ 29, 000 ksi. Fig. E4.11a I ¼ 300 in.4 346 4 Statically Determinate Plane Frames Determine: The horizontal and vertical deflections and the rotation at point C, the tip of the cantilever segment. Solution: We start by evaluating the moment distribution corresponding to the applied loading. This is defined in Fig. E4.11b. We next determine the virtual moment distributions corresponding to uc, vc, and θc are defined in Figs. E4.11c, E4.11d, and E4.11e, respectively. We note that we take δP to be either a unit force (for displacement) or a unit moment (for rotation). Fig. E4.11b M(x) Fig. E4.11c δM(x) for uc Fig. E4.11d δM(x) for vc 4.6 Deflection of Frames Using the Principle of Virtual Forces 347 Fig. E4.11e δM(x) for θc We divide up the structure into two segments AB and CB and integrate over each segment. The total integral is given by ð ð X ð M M M 6M ds ¼ 6M dx1 þ 6M dx2 EI AB EI CB EI members s The expressions for uc, vc, and θc are generated using the moment distributions listed above. δ 10 EIuC ¼ 621:6Þð10 þ x1 Þdx1 ¼ 1080 kip ft3 0 108012 3 uC ¼ ¼ 0:2145 in. ! 29, 000ð300 δ 10 δ 10 1:2 2 x dx2 þ 2x2 ¼ 1490 kip ft3 EIvC ¼ 6ð21:6Þð6dx1 þ 2 2 0 0 vC ¼ 1490ð12Þ3 ¼ 0:296 in. # 29, 000ð300Þ ð 6 10 1:2 2 x dx2 ¼ 259 kip ft2 EIθC ¼ ð21:6Þð1Þdx1 þ 2 0 0 ¼ 0:0043 rad clockwise 29, 000ð300Þ 348 Computation of Deflection Given: The structure shown in Fig. E4.12a. Solution: We begin first by evaluating the moment distribution corresponding to the applied loading. This is defined in Fig. E4.12b. Fig. E4.12b M(x) The virtual moment distributions corresponding to uc and vc are shown in Figs. E4.12c and E4.12d. Fig. E4.12c δM for uC 4.6 Deflection of Frames Using the Principle of Virtual Forces 349 Fig. E4.12d δM for vC Fig. E4.12e δM for θB We express the total integral as the sum of three integrals. ð ð X ð M M M 6M ds ¼ 6M dx1 þ 6M dx2 þ 6M dx3 EI AB EI DB EI members s EI CD The corresponding form for uC is δ 10 δ 10 δ 10 EIuC ¼ ð20 x2 Þð10x2 Þdx1 þ ð10 þ x2 Þdx1 þ ð20 x2 Þð23x3 x23 dx3 0 0 0 δ 10 δ 10 x2 ðþ23x3 x23 dx3 0 0 30x22 2 600x2 0 þ 3 43 ¼ 32, 333 kip ft3 0 3 uC ¼ 32, 333ð12Þ3 ¼ 1:93 in. ! 029; 000Þð900Þ 4 Statically Determinate Plane Frames The corresponding form for vC is δ 10 δ 10 20 x2 þ 23x3 x23 dx3 3 0 0 0 10 23x3 1 4 20 3 ¼ 2x31 0 þ j600x2 j 0 þ x 10 þ 3 8 þ 3 ¼ 18, 667 kip ft3 6 0 3 uD ¼ 18, 667ð12Þ3 ¼ 1:23 in. ! 029; 000Þð900Þ Lastly, we define θB (Fig. E4.12e) δ 10 ð 10 δ 10 20 x2 EIθB ¼ þ 23x3 x23 dx3 ð1Þdx1 þ 30x22 x23 3 0 0 ¼ 30x2 þ 3 0 þ 4 20 ¼ 1, 667 kip ft2 0 θB ¼ 1:667ð12Þ2 ¼ 0:0059 ¼ 0:0059 rad 029; 000Þð900Þ The minus sign indicates the sense of θB is opposite to the initial assumed sense. θB ¼ 0:0059 rad clockwise Example 4.13 Computation of Deflection Given: The steel structure shown in Fig. E4.13a, E4.13b, and E4.13c. Take I ¼ 43l c , hC ¼ 4 m, Lb ¼ 5 m, P ¼ 40 kN, and E ¼ 200 GPa. Determine: The value of IC required to limit the horizontal displacement at C to be equal to 40 mm. Fig. E4.13a 4.6 Deflection of Frames Using the Principle of Virtual Forces 351 Solution: We divide the structure into two segments and express the moments in terms of the local coordinates x1 and x2. Fig. E4.13b M(x) Fig. E4.13c δM for uC We express the total integral as the sum of two integrals. ð ð X ð M M M 6M ds ¼ 6M dx1 þ 6M dx2 EI AB EI CB EI members s The corresponding expression for uC is ð ð 1 hc 1 Lb hc Lb Lb uC ¼ Ph½x1 Pðx1 Þdx1 þ P x2 x2 dx2 EI c 0 EI ð hc 1 Lb Lb uC ¼ P EI c 0 hc Phc 2 þ δ 0 Lb P hc 0 Ph3c PL3b hc 2 Ph2c hc Lb uC ¼ þ 3EI c 3EI b 3EI c 3 EI c þ 40 k P Lb uC ¼ ð3EI c Þ 6 PL3b 1 3EI c 3EI I c Lb 1 c Lb PL ¼ ¼ þ 3 3 Solving for Ic, the IC required is determined with Ph2c hc Lb 404000Þ2 4000 3000 þ þ Ic ¼ ¼ 3uC 3 ð3Þ40 Ic ¼ 30200Þ ·∴Ic ¼ 167ð10Þ6 mm4 Example 4.14 Computation of Deflection—Non-prismatic Member Given: The non-prismatic concrete frame shown in Fig. E4.14a and E4.14b. Fig. E4.14a Non-prismatic frame Assume hC ¼ 12 ft, Lb ¼ 10 ft, P ¼ 10 kip, and E ¼ 4000 ksi. Consider the member depths (d) to vary linearly and the member widths (b) to be constant. Assume the following geometric ratios: dA/dB, ¼ 1, dC/dB, ¼ 1. Use the principle of Virtual Forces. Fig. E4.14b Cross section depths Determine: (a) A general expression for the horizontal displacement at C (uC). (b) Use numerical integration to evaluate uC as a function of dAB, dB,. (c) The value of dAB,0 for which uC ¼ 1.6 in. Solution: Part (a): The member depth varies linearly. For member AB, x1 x1 x1 dAB, 1 dð x1 Þ ¼ dA þ ðdB dA Þ ¼ dB, 1 þ dA, 1 dB, 0 ¼ dB, 0 ¼ 1 þ 0gAB þ hc hc hc hc I AB ¼ I AB, 0 ðgAB þ 3 Similarly, for member BC x2 dCB, 0 x1 dAB, 0 ¼ 0 þ 1 þ 0gCB Lb dCB, 0 Lb I CB ¼ I CB, 0 1 þ 0gCB 1 Lb Lb 3 353 354 4 Statically Determinate Plane Frames We express the moment in terms of the local coordinates x1 and x2. M0x1 Þ ¼ Px1 0 < x1 < hc hc M0x2 Þ ¼ P x2 Lb 0 < Lb δM6x1 Þ ¼ x1 δM6x2 Þ ¼ x2 Lb 0 < x2 < Lb Fig. E4.14c M(x) Fig. E4.14d δM for uC 1 uC ¼ E ð hc 0 1 1 Pð x1 Þðx1 Þdx1 1 I AB E I Lb 0 hc hc P x2 x2 dx2 1 CB I CB Lb Lb 1 Substituting for IAB and ICB and expressing the integral in terms of the dimensionless values ξ1 ¼ x1/hc and ξ2 ¼ x2/Lb, the expression for uC reduces to 4.6 Deflection of Frames Using the Principle of Virtual Forces P3hc h3 uC ¼ EI AB, 0 3 ð ξ1 ξ1 dξ1 3 2 1 ð 1 þ gAB ξ1 Þ3 0 0 þ Lb 2 ð ξ2 ð1 ξ2Þ 2 dξ2 þ 1 þ gCB ð1 ξ2Þ 3 EI CB, 0 gCB þ gAB Þ3 1 ¼ 0 taking g ¼ 0, 1 ¼ 3 Part (b): We set the gAB ¼ gCB ¼ g. Then, the problem reduces to evaluating the following integrals: 2 ð1 ξ1 dξ1 ξ2 ð1 ξ2 Þ2 dξ2 and ð1 ð ξ 2 Þ 3 ð 1 þ gðξ1 Þ3 dξ1 1 ð1 þ gð1 ξ2Þ Þ3 0 0 I1 ¼ 2 ð1 þ gξ1 Þ3 0 ξ1 ¼ 2 þ ξ1 2 ð1 gðξ1 Þ3 ð1 þ g ξ1 ξ2 þ gξ1 Þ3 ξ1 ¼ 2 d ξ1 0 I2 ¼ 2 The Combination ½ ð1 þ gξ1 Þ3 ξ1 dξ1 þ 1 ð 1 þ gð1 ξ2 Þ3 1 þ gð1 ξ2 Þ3 dξ1 þ 1 ¼ 2 δð1 þ gξ2 Þ3 ξ2 dξ2 We compute these values using the trapezoidal rule. Results for different interval sizes are listed below. N 10 20 25 30 J1 0.0682 0.0682 0.0682 0.0682 J2 0.1329 0.1329 0.1329 0.1329 Next, we specify the inertia terms I AB, 0 ¼ I CB, 0 ¼ I BC, 0 ¼ 1 12 b 12 IBC, 0 ¼ 12 b d3 CB, 0 ¼ 12 ð b d CB, 0 ð 12 b0 dCB, 0 3 12 IAB,0 ¼ 1 12 b dCB, 0 3 12 IAB,0 ¼ (3/4)ICB,0 4 IAB,0 ¼ 4 Lb Part(c): Setting uC ¼ 1.86 and solving for IAB,0 leads to ( ) 2 P h3 c 3 uC ¼ ¼ h3c J 1 þ Lb3 þδJ 2 þ EI AB 0 4 Lb Part) ¼ 2 P h3 c 3 uC ¼ þ h3c J 1 þ þδLb þ 3 J 2 þ 3 EI AB, 0 ¼ 4 f2 h3 c 3 Lb Finally, dCB, 0 ¼ 0 ¼ 2 0 , 0 ¼ 4 f2 h3 c 3 Lb ð 1 þ ð12 Þ J 2 10:98 in: 1 dCB, 0 ¼ 1 dCB, 0 ¼ 2 0 ¼ 40:981:2 f2 h3 c 3 Lb Ic ¼ 2 Þ12 ¼ 0 Finally, dAB, 0 ¼ 1 dCB, 0 ¼ 1 ¼ 6:81 ð12, 0 g1=4 ¼ 10:98 in: 1=3 dCB, 0 ¼ 1:4 dAB, 0 ¼ 1:in: 356 4.7 Deflection Profiles: Plane Frame Structures 4 Statically Determinate Plane Frames Deflection Profiles: Plane Frame Structures In Chap. 3, we showed how to sketch the deflected shapes of beams given the bending moment distributions. The principle of Virtual Forces can be applied to compute the displacement at specific points. If one is more interested in the overall displacement response, then it is necessary to generate the displacement profile for the frame. We dealt with a similar problem for beams. Now, we extend these procedures to plane frame structures. We basically follow the same approach in this case, except now we consider member displacements, as shown in Fig. 4.26. In order to establish the deflection profile for the frame, one needs to construct the profile for each member, and then join up the individual shapes such that the displacement restraints are satisfied. We followed a similar strategy for planar beam-type structures; however, the process is somewhat more involved for plane frames. Consider the portal frame shown in Fig. 4.27. Bending does not occur in AB and CD since the moment is zero. Therefore, these members must remain straight. However, BC bends into a concave shape. The profile consistent with these constraints is plotted below. Note that B, C, and D move laterally under the vertical loading. Suppose we convert the structure into the 3-hinge frame (Fig. 4.28). Now, the moment is negative for all members. In this case, the profile is symmetrical. There is a discontinuity in the slope at E because of the moment release. Fig. 4.26 Momentcurvature relationship Fig. 4.27 Portal frame. (a) Loading. (b) Bending moment. (c) Deflection profile Fig. 4.28 3-hinge frame. (a) Loading. (b) Bending moment. (c) Deflection profile 4.7 Deflection Profiles: Plane Frame Structures 357 Gable frames are treated in a similar manner. The deflection profiles for simply supported and 3-hinge gable frames acted upon by gravity loading are plotted below (Figs. 4.29 and 4.30). Fig. 4.29 Simply supported gable frame. (a) Loading. (b) Bending moment. (c) Deflection profile Fig. 4.30 3-Hinge gable frame. (a) Loading. (b) Bending moment. (c) Deflection profile The examples presented so far have involved gravity loading. Lateral loading is treated in a similar way. One first determines the moment diagram and then establishes the curvature patterns for each member. Lateral loading generally produces lateral displacements as well as vertical displacements. Typical examples are listed below (Figs. 4.31, 4.32, and 4.33). Fig. 4.31 Portal frame. (a) Loading. (b) Bending moment. (c) Deflection profile Fig. 4.32 3-Hinge frame. (a) Loading. (b) Bending moment. (c) Deflection profile 358 4 Statically Determinate Plane Frames Fig. 4.33 3-Hinge gable frame. (a) Loading. (b) Bending moment. (c) Deflection profile 4.8 Computer-Based Analysis: Plane Frames When there are multiple loading conditions, constructing the internal force diagrams and deflection profiles is difficult to execute manually. One generally resorts to computer-based analysis methods specialized for frame structures. The topic is discussed in Chap. 12. The discussion here is intended to be just an introduction. Consider the gable frame shown in Fig. 4.34. One starts by numbering the nodes and members, and defines the nodal coordinates and the nodal constraints. For plane frames, there are two translations and one rotation at each node (two translations and one rotation). Therefore, there are three possible displacement restraints at a node. For this structure, there are two support nodes, nodes 1 and 5. At node 1, the X and Y translations are fully restrained, i.e., they are set to zero. At node 5, the Y translation is fully restrained. Fig. 4.34 Geometry and loading Next, information related to the members, such as the cross-sectional properties (A, I), loading applied to the member, and releases such as internal moment releases are specified. Finally, one specifies the desired output. Usually, one is interested in shear and moment diagrams, nodal reactions and displacements, and the deflected shape. Graphical output is most convenient for visualizing the structural response. Typical output plots for the following cross-sectional properties I1 ¼ 100 in.4, I2 ¼ 1000 in.4, I3 ¼ 300 in.4, E ¼ 29,000 ksi, A1 ¼ 10 in.2, A2 ¼ 88 in.2, and A3 ¼ 22 in.2 are listed in Fig. 4.35. 4.8 Computer-Based Analysis: Plane Frames Fig. 4.35 Graphical output for structure defined in Fig. 4.34. (a) Displacement profile. (b) Bending moment, M. (c) Shear, V. (d) Axial force, F. (e) Reactions 359 360 4.9 4 Statically Determinate Plane Frames Plane Frames: Out of Plane Loading Plane frames are generally used to construct three-dimensional building systems. One arranges the frames in orthogonal planes to form a stable system. Figure 4.36 illustrates this scheme. Gravity load is applied to the floor slabs. They transfer the load to the individual frames resulting in each frame being subjected to in-plane loading. This mechanism is discussed in detail in Chap. 15. Our interest here is the case where the loading acts normal to the plane frame. One example is the typical highway signpost shown in Fig. 4.37. The sign and the supporting member lie in a single plane. Gravity load acts in this plane. However, the wind load is normal to the plane and produces a combination of bending and twisting moment in the column. We deal separately with the bending and torsion problems and then superimposes the results. The typical signpost shown in Fig. 4.37 is statically determinate. We sketch the free body diagram shown in Fig. 4.38. The wind load acting on the sign produces bending and twisting moment in the column. We use a double-headed arrow to denote the torsional moment. Suppose the Y displacement at C is desired. This motion results from the following actions: Member BC bends in the X Y plane 3 b Pw 2 vC ¼ 3EI 2 Member AB bends in the Y Z plane and twists about the Z axis Pw b2 h Pw b 3 vB ¼ ðBÞ ¼ GJ 3EI 1 where GJ is the torsional rigidity for the cross section. Fig. 4.36 A typical 3-D system of plane frames 4.9 Plane Frames: Out of Plane Loading Fig. 4.37 Signpost structure. Fig. 4.38 Free body diagrams 361 362 4 Statically Determinate Plane Frames Fig. 4.39 Plane grid structure Node C displaces in the out of plane direction at B 2 b h vC ¼ Pw ðBbz ¼ 2 2 GJ Summing the individual contributions leads to 3 b h3 b2 h vC total ¼ Pw þ þ 24EI 2 3EI 1 4GJ Another example of out-of-plane bending is the transversely loaded grid structure shown in Fig. 4.39. The members are rigidly connected at their ends and experience, depending on their orientation, bending in either the X Z plane or the Y Z plane, as well as twist deformation. Plane grids are usually supported at their corners. Sometimes, they are cantilevered out from one edge. Their role is to function as a plate-type structures under transverse loading. Plane grids are usually statically indeterminate systems. Manual calculations are not easily carried out for typical grids so one uses a computer analysis program. This approach is illustrated in Chap. 10. 4.10 Summary 4.10.1 Objectives • To develop criteria for static determinacy of planar rigid frame structures. • To present an analysis procedure for statically determinate portal and pitched roof plane frame structures subjected to vertical and lateral loads. • To compare the bending moment distributions for simple vs. 3-hinged portal frames under vertical and lateral loading. • To describe how the Principle of Virtual Forces is applied to compute the displacements of frame structures. • To illustrate a computer-based analysis procedure for plane frames. • To introduce the analysis procedure for out-of-plane loading applied to frames. 4.11 Problems 363 4.10.2 Key Concepts • A planar rigid frame is statically determinate when 3 m + r ¼ 3j, where m is the number of members, r is the number of displacement restraints, j is the number of nodes, and n the number of releases. • A planar A-frame is statically determinate when 3 m ¼ r + 2np, where np is the number of pins, m is the number of members, and r is the number of displacement restraints. • The Principle of Virtual Forces specialized for frame structures has the general form of δP ¼ X δ M F δM þ δF ds EI AE For low-rise frames, the contribution of the axial deformation term is negligible. • The peak moments generated by lateral loading are generally less than for simple portal frames. 4.11 Problems For the plane frames defined in Problems 4.1–4.18, determine the reactions, and shear and moment distributions. Problem 4.1 364 Problem 4.2 Problem 4.3 4 Statically Determinate Plane Frames 4.11 Problems Problem 4.4 Problem 4.5 Problem 4.6 365 366 Problem 4.7 Problem 4.8 Problem 4.9 4 Statically Determinate Plane Frames 4.11 Problems Problem 4.10 Problem 4.11 Problem 4.12 367 368 Problem 4.13 Problem 4.14 Problem 4.15 4 Statically Determinate Plane Frames 4.11 Problems 369 Problem 4.16 Problem 4.17 Problem 4.18 For the frames defined in Problems 4.19–4.26, determine the bending moment distributions. 370 Problem 4.19 Problem 4.20 Problem 4.21 4 Statically Determinate Plane Frames 4.11 Problems Problem 4.22 Problem 4.23 Problem 4.24 371 372 4 Statically Determinate Plane Frames Problem 4.25 Problem 4.26 For the A-frames defined in Problems 4.27–4.29, determine the reactions and bending moment distributions. Problem 4.27 4.11 Problems Problem 4.28 Problem 4.29 373 374 4 Statically Determinate Plane Frames Problem 4.30 Determine the horizontal deflection at D and the clockwise rotation at joint B. Take E ¼ 29,000 ksi and I ¼ 200 in.4 Use the Virtual Force method. Problem 4.31 Determine the value of I required to limit the horizontal displacement at D to 2 in. Use the Virtual Force method. Problem 4.32 Determine the vertical deflection at C to 30 mm. Take E ¼ 200 GPa. Use the Virtual Force method. 4.11 Problems 375 Problem 4.32 Determine the value of I required limiting the horizontal displacement at C to 30 mm. Take E ¼ 200 GPa and I ¼ 60(10)6 mm4. 376 4 Statically Determinate Plane Frames Problem 4.34 Determine the horizontal deflection at D and the rotation at joint B. Take E ¼ 200 GPa and I ¼ 40(10)6 mm4. Use the Virtual Force method. Problem 4.35 Determine the displacement at the roller support C. Take E ¼ 29,000 ksi and I ¼ 100 in.4 Use the Virtual Force method. 4.11 Problems 377 Problem 4.36 Determine the horizontal deflection at C and the rotation at joint B. Take E ¼ 29,000 ksi and I ¼ 160 in.4 Use the Virtual Force method. Problem 4.37 Determine the horizontal deflection at C and the vertical deflection at C. Take E ¼ 29,000 ksi and I ¼ 160 in.4 Use the Virtual Force method. Problem 4.38 Determine the horizontal deflection at C. I ¼ 100(10)6 mm4 and E ¼ 200 GPa. Use Virtual Force method. 378 4 Statically Determinate Plane Frames Problem 4.39 Sketch the deflected shapes. Determine the vertical and horizontal deflection at A. Take I ¼ 240 in.4, E ¼ 29,000 ksi, and h ¼ 2b ¼ 10 ft. Problem 4.40 Determine the deflection profile for member DBC. Estimate the peak deflection. Use computer software. Note that the deflection is proportional to 1/EI. 4.11 Problems 379 Problem 4.41 Consider the pitched roof frame shown below and the loadings defined in cases (a)–(f). Determine the shear and bending moment distributions for each case. Use a computer software system. Take I ¼ 10,000 in.4 (4160(10)6 mm4), E ¼ 30,000 ksi (200 GPa). 380 4 Statically Determinate Plane Frames Problem 4.42 Consider the frame shown below. Determine the required minimum I for the frame to limit the vertical deflection at E to 15 mm. The material is steel. Use computer software. Problem 4.43 Consider the frame shown below. Determine the required minimum I for the frame to limit the vertical deflection at E to 15 mm. The material is steel. Use computer software. Problem 4.44 Consider the triangular rigid frame shown below. Assume the member properties are constant. I ¼ 240 in.4, A ¼ 24 in.2 and E ¼ 29,000 ksi. Use computer software to determine the axial forces and end moments for the following range of values of tan θ ¼ 2 h/L ¼ 0.1, 0.2, 0.3, 0.4, 0.5 Compare this solution with the solution based on assuming the structure is an ideal truss. Reference 381 Problem 4.45 Consider the structure consisting of two members rigidly connected at B. The load P is applied perpendicular to the plane ABC. Assume the members are prismatic. Determine θy at point C (defined as θc on the figure). Problem 4.46 Members AB, BC, and CD lie in the X Y plane. Force P acts in the Z direction. Consider the cross-sectional properties to be constant. Determine the Z displacement at D. Take pffiffiffiffiffi LAB ¼ L, LBC ¼ L, LCD ¼ L, 2 Reference 1. Reference 1. Tauchert TR. Energy principles in structural mechanics. New York: McGraw-Hill; 1974. 5 Cable Structures Abstract Historically, cables have been used to construct bridges, tents, and other structures. In this chapter, we first examine how the geometry of a cable is related to the loading that is applied to it. We treat concentrated loadings first and then incorporate distributed loadings leading up to a theory for continuously loaded inclined cables. We also analyze the effect of temperature on the cable geometry. Lastly, we develop an approximate formula for estimating the stiffness of a cable modeled as an equivalent straight member. This modeling strategy is used when analyzing cablestayed structures. 5.1 Introduction A cable is a flexible structural component that offers no resistance when compressed or bent into a curved shape. Technically, we say a cable has zero bending rigidity. It can support only tensile loading. The first cables were made by twisting vines to form a rope-like member. There are many examples of early suspension bridges dating back several thousand years. With the introduction of iron as a structural material, cable-type members were fabricated by connecting wrought iron links. Figure 5.1 shows an example of an iron link suspension bridge, the Clifton Suspension Bridge near Bristol, England built in 1836–1864 and designed by Isambard Brunel. When high-strength steel wires became available, steel replaced wrought iron as the material of choice for cables. Modern cables are composed of multiple wires (up to 150 wires) clustered in a circular cross-section and arranged in a parallel pattern, as illustrated in Fig. 5.2. This arrangement is used for cable-stayed bridges and suspension bridges. The steel cable is normally coated with a protective substance such as grease and wrapped or inserted in a plastic sheathing. One of the most famous suspension bridges was the Brooklyn Bridge built in 1870–1883 and designed by John Roebling and Wilhelm Hildebrandt. John Roebling also invented and perfected the manufacture of steel wire cable which was used for the bridge. At the time of completion, the total length of the Brooklyn Bridge was 50 % greater than any existing suspension bridge, an extraordinary advancement in bridge engineering (Fig. 5.3). # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_5 383 384 5 Cable Structures Fig. 5.1 Clifton Suspension Bridge, England Fig. 5.2 Cable-strand arrangements Cable nets are used as the primary structural elements for long-span horizontal roof structures. Figure 5.4 shows a single-layer cable net structure with a double-curved saddle-shaped surface designed by Schlaich Bergermann and partners for a stadium in Kuwait. Cable-stayed systems employ cables fabricated from ultra high-strength steel to allow for the high level of tension required for stiffening the bridge deck. Over the past few decades, the cable-stayed bridge concept has emerged as the predominant choice for main spans up to about 1000 m, replacing the conventional truss structural system. Figure 5.5 shows the Normandy Bridge, with a main span of 856 m. Built in 1995, it held the record for the longest main span until 1999, when it was exceeded by the Tatara Bridge in Japan. 5.1 Introduction Fig. 5.3 Brooklyn Bridge, USA Fig. 5.4 Doubly curved single-layer cable net, Kuwait 385 386 5 Cable Structures Fig. 5.5 Normandy Bridge, France 5.2 Cables Subjected to Concentrated Loads 5.2.1 Horizontal Cables Suppose we conduct the following experiment shown in Fig. 5.6. We start with a horizontal straight cable that is pin connected at A, supported with a roller support at B, and tensioned with a force H. We then apply a concentrated force, P, at mid-span. The cable adopts the triangular shape shown under the action of P. Two questions are of interest. Firstly, why is a triangular shape produced? Secondly, how is the 5.2 Cables Subjected to Concentrated Loads 387 Fig. 5.6 Transverse loading on pretensioned cable. (a) Axial load. (b) Transverse load applied. (c) Free body diagram. (d) Free body diagram of cable segment downward vertical displacement at mid-span related to P and H? Historically, the term "sag" is used to describe the vertical motion of a cable. We answer these questions by noting that the moment at any section along the length of the cable must be zero since a cable has no resistance to bending. Summing moments about B X M ¼ RA L P at B L P ¼ 0) RA ¼ " 2 2 Next, we consider the free body diagram for the arbitrary segment shown in Fig. 5.6d. Setting the moment at x equal to zero leads to an expression for the sag, v(x). X Mat x ¼ P x HvðxÞ ¼ 0 2 vðxÞ ¼ P x 2H 0 5 x 5 L=2 ð5:1Þ þ ð5:2Þ Finally, evaluating v(x) at x ¼ L/2 results in an equation relating vC and P. vC ¼ PL 4H ð5:3Þ The relationship between vC and H is plotted below in Fig. 5.7. Usually, one specifies H and determines vC. However, there are cases where one specifies vC and determines the required value of H. In general for cable systems, one needs to specify either a force or a sag in order to define the solution. 388 5 Cable Structures Fig. 5.7 Relationship between vc and H The tension in the cable is given by pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 2ffi 2ffi P H2 þ 2 The tension in the cable is related to the sag by tan θ ¼ vC P=2 ¼ L=2 H leads to the expression T ¼ H 1 þ 4 vC L The angle of inclination of the cable is related to the sag by tan θ ¼ vC P=2 ¼ L=2 leads to the approximate form T ¼ ð5:4Þ H ¼ H 1 þ tan 2 θ ¼ H cos θ When θ is small, T is approximately equal to H. ð5:5Þ ð5:6Þ 5.2 Cables Subjected to Concentrated Loads 389 Fig. 5.8 Moment distribution for simply supported beam M0(x) Equation (5.1) can also be written as vðxÞ ¼ M0ðxÞ H ð5:7Þ where M0(x) is the moment due to the transverse loading acting on a simply supported beam spanning between the cable supports. We extend this reasoning to a cable subjected to multiple concentrated loads. Figure 5.8 shows this distribution. We express (5.1) as M0 ðxÞ vðxÞ ¼ H ð5:8Þ We interpret this result as follows. The shape of the vertical sag of the cable from the horizontal chord is a scaled version of the moment diagram for the transverse loading acting on a simply supported beam spanning between A and B. Then, the expression for the sag can be written as v ðx Þ ¼ M0 ðxÞ H ð5:8Þ We interpret this result as follows. The shape of the vertical sag profile is piecewise linear, with peak values at the points of application of the concentrated loads. It follows from (5.8) that the shape of the cable is also piecewise linear. Figure 5.9a illustrates this result. The moment diagram for a set of concentrated loads is piecewise linear, with peak values at the points of application of the loads. It follows that the shape of the cable is also piecewise linear. One generates M0(x), the corresponding shear V0(x), and the displacement v, and the tension T. Details are listed in Fig. 5.9b–d. Note that one has to specify either H or a vertical coordinate (vC or vD) in order to compute the shape. T¼ qffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi V 20 þ H2 ¼ H cos θ Example 5.1 Cable with Two Concentrated Loads Given: The cable and loading shown in Fig. E5.1a. 390 5 Cable Structures Fig. 5.9 Cable with two concentrated loads. (a) Loading. (b) V0(x). (c) M0(x). (d) Cable tension computation 5.2 Cables Subjected to Concentrated Loads 391 Determine: The shape corresponding to this loading. Assume (a) vD ¼ 6 ft (b) vD ¼ 12 ft Fig. E5.1a Cable and loading Solution: First, we find the vertical reactions and construct the shear diagram V0(x) and moment diagram, M0(x), treating chord AB as a simply supported beam acted upon by the two vertical forces (Fig. E5.1b). Fig. E5.1b Simply supported beam results The downward vertical sags from the chord AB are given by (5.8) ð# þÞ vðxÞ ¼ M0 ðxÞ H Then we need the horizontal force, H. (a) Taking vD ¼ 6 ft results in 6½ ¼ 610 H ¼ 101:67 kip 392 5 Cable Structures Fig. E5.1c Sag profile for vD ¼ 6 ft Once the value of H is known, one can determine the tension in the various segments using (5.9). The final results are listed in Fig. E5.1c. vC ¼ 3:85 ft 101:67 vC ¼ 3:85 ft The remaining sags are 364 ¼ 3:58 ft 101:67 391:6 vE ¼ 3:85 ft 101:67 The tension in the various segments (Fig. E5.1d) are T AB ¼ pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 18:22 þ 101:672 ¼ 103:3 kip pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi T CD ¼ 8:22 þ 101:672 ¼ 102 kip pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi T DE ¼ 5:82 þ 101:672 ¼ 101:8 kip pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi T DE ¼ 17:82 þ 101:672 ¼ 103:2 kip Fig. E5.1d Force decomposition (b) Taking vD ¼ 12 ft results in 610 ¼ 50:83 kip 12 364 vC ¼ 7:7 ft 50:83 H¼ and T AC ¼ 54 kip T CD ¼ 51:5 kip T DE ¼ 51:16 kip T EB ¼ 53:85 kip 5.2 Cables Subjected to Concentrated Loads 393 The sag profile is plotted below (Fig. E5.1e) Fig. E5.1e Sag profile for vD ¼ 12 ft Note that increasing the prescribed value of vD decreases the cable tension. 5.2.2 Inclined Cables Fig. 5.10 Inclined cable with two concentrated loads Consider the cable shown in Fig. 5.10. We follow the same approach except that now we measure the sag with respect to the inclined chord. Using the free body diagram shown in Fig. 5.11. Note that these results are identical to the results for the case of the horizontal chord. The reactions generated by the horizontal cable force, H, are determined by equating the moment at A to zero. Setting the moment equal to zero leads to yB ¼ h ¼ HyðxÞ Þ Then, the moment at x, X M ¼ 0 þ HvðxÞ Þ vðxÞ ¼ H0 ðxÞ H Lh ¼ yB x ¼ yðxÞ Þ yðxÞ Hv0ðxÞ ¼ M0 ðxÞ ¼ H M0 0xÞ Þ vðxÞ H ¼ M0 ðxÞ H Note that the solution for v(x) is identical to the results for the horizontal chord case. Equation (5.10) is identical to (5.8) the sag measured from the inclined chord. Therefore, the approach followed for the horizontal cable except that now one measures the sag from the inclined chord. 394 Fig. 5.11 Simply supported beam results. (a) Vertical loading. (b) V0(x) diagram. (c) M0(x) diagram. Fig. 5.12 Reactions due to H 5 Cable Structures 5.2 Cables Subjected to Concentrated Loads Example 5.2 Analysis of an Inclined Cable Given: The inclined cable and loading shown in Fig. E5.2a. 5 Cable Structures 5.2 Cables Subjected to Concentrated Loads 395 Determine: The sag of the cable. Assume vD ¼ 6 ft. Fig. E5.2a Inclined geometry Solution: According to the theory presented above, the sag with respect to the inclined chord is given by þ# vðxÞ¼ M0 ðxÞ H where M0(x) is the simply supported beam results Then, vC ¼ 364 H vD ¼ 610 H vE ¼ 6 ft. Then, the value of H follows from H¼ M0D 610 ¼ ¼ 101:67 kip vD 6 396 5 Cables Structures Finally, the values of sag at C and E are 364 ¼ 3:58 ft 101:67 391:6 vE ¼ 3:85 ft 101:67 To determine the tension, we need to determine the vertical component. Fig. E5.2b Sag profile and vertical forces Note that the maximum tension is in segment AC. The vertical reactions are shown in Fig. E5.2b). Fig. E5.2c Vertical reactions due to H We sum the vertical forces at each node. The vertical forces at each node are HyB 101:67ð4Þ ¼ 3:7 kip ¼ 110 L The total vertical force at A is 18.2 kip. The total vertical force at B is 17.8 kip. Fig. E5.2d Vertical shear are shown in Fig. E5.2c. We sum the forces for the various segments using the final values of V. T AB ¼ pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 21:92 þ 101:672 ¼ 104 kip pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi T CD ¼ 11:92 þ 101:672 ¼ 102:4 kip pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi T EA ¼ 14:092 þ 101:672 ¼ 102:6 kip T AC ¼ 5 Cables Structures 5.3 Cables Subjected to Distributed Loading 397 pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi T DE ¼ 2:12 þ 101:672 ¼ 101:7 kip pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 5.3 Cables Subjected to Distributed Loading 5.3.1 Horizontal Cable: Uniform Loading per Horizontal Projection Fig. 5.13 Cable with a uniformly distributed load We consider next the cable system shown in Fig. 5.13. The cable supports a horizontal platform, which in turn, supports a uniformly distributed load. We represent the action of the closely spaced vertical hangers on the cable as a uniform downward loading per unit horizontal length. The self-weight of the cable, which is usually small in comparison to the applied loading, is neglected. Following the procedure described in the previous section, the sag of the cable with respect to the horizontal chord is an inverted scaled version of the moment diagram for the simply supported beam spanning between the two end supports. Using the free body diagram shown in Fig. 5.14. The slope, tan θ, and T are given by. Then M0 ðxÞ vðxÞ ¼ H L 2 2 L 2 2 2 Lx x2 M0ðxÞ ¼ wL x2 ¼ w L x x2 2 2 ¼ w 2H L x2 2 ¼ 2H dv 1 dM0 ðxÞ ¼ H dx ¼ 1 H dv0ðxÞ ¼ dx ¼ w 2H ðL 2xÞ tan θ ¼ dv dx ¼ 1 dM0 ðxÞ ¼ w H dx 2H T¼ V0 ¼ cos θ H ð5:9Þ It follows that the shape due to a uniform load is parabolic and the maximum sag occurs at mid-span, point c. vC ¼ vðL=2Þ ¼ wL2 8H ð5:10Þ 398 5 Cable Structures Fig. 5.14 Horizontal cable. (a) Simply supported beam. (b) Free body diagram. (c) Simply supported beam. (d) Cable Structures 5.3 Cables Subjected to Distributed Loading 399 Solving for H: We note that the maximum value of T occurs at x ¼ L/2. Then, specializing (5.9) to the value of x leads to the value of H which is the same as (5.10). Then, H ¼ wL2 8vC ð5:11Þ The tension is related to H by: Tmax ¼ H cos θ Note that the maximum value of θ occurs at x ¼ 0 and x ¼ L. w0 L L 015Þ30Þ2 ¼ tan θ ¼ 2H tan θmax ¼ ð12 1 4 L 4 vC L L x3 0 x 4 2 3L 1 dM0 xÞ ¼ 2 H tan θmax ¼ 4vC L Then, T max ¼ H 1 þ 16 L 2 ¼ 1 H cos θmax vC Using Table 5.1 T max ¼ H vC ¼ 0:5 vC Example 5.3 Cable with a Uniformly Distributed Load Given: The cable and loading shown in Fig. E5.3a. L ¼ 1 w0 L 015Þ30Þ2 T ¼ 0 1 2 ¼ 4 0 2 3 tan θmax 1 4 0:5ð30Þ ¼ 3 4 ¼ 0:75 Fig. E5.3a Cable geometry H ¼ w0 L2 8vC ¼ 15ð30Þ2 8ð3Þ ¼ 562:5 kN Tmax ¼ H sqrt1 þ tan2 θ ¼ 562:5 ¼ 375 kN ¼ 1 þ 16ð0:1Þ2 ¼ 606 kN The tension is given by: T¼ H cos θ Tmax ¼ 1 þ 16 vC L 2 H Suppose the cable is subjected to a uniform load per unit arc length. 5.3.2 Inclined Cable: Uniform Loading per Horizontal Projection Consider the cable shown in Fig. 5.15. Suppose the cable is subjected to an arbitrary loading. We define the shape by the function y(x). Since the cable has no bending rigidity, the moment at any point on the cable must be zero. The derivation is similar to the case of the inclined cable. Using the free body diagram shown in Fig. 5.15, The results reduce to an expression for the sag of the cable must adjust itself so that the resultant moment due to the vertical load and H vanishes at all points along the cable. Summing moments at x leads to X Mat x ¼ M0 ðxÞ þ H yB x ¼ HvðxÞ þ y ðx Þ þ Hy ðxÞ ¼ 0 ¼ 0 vðxÞ ¼ M0 ðxÞ H þ yB x L Since M0(xÞ ¼ M0 ðxÞ ¼ þ Hvðx Þ L HyB ¼ HyB x ¼ 0 ð5:12Þ 400 5 Cable Structures Fig. 5.15 Inclined cable geometry—arbitrary loading. (a) Geometryarbitrary loading. (b) Simply supported beam results. (c) Reactions due to horizontal force. H We note from Fig. 5.15 that y ðx Þ þ vðxÞ ¼ yB x L Then, yðx Þ ¼ yB x L H M0 ðxÞ ¼ H yB x L ¼ yB x L H M0 0xÞ L Then, expressing (5.11) and (5.12) leads to an expression for the sag, vðxÞ ¼ M0 ðxÞ H ð5:13Þ We observe that the solution for the sag is identical to the result that we obtained for the horizontal chord orientation except that now we measure the sag from the inclined chord. The solution is also similar to the case of a set of concentrated loads. 5.3 Cables Subjected to Distributed Loading 401 Fig. 5.16 Cable geometry—lowest point. Noting (5.11), (5.12) and (5.13) and the uniform loading, M0(x) is parabolic and the maximum sag occurs at mid-span. y ¼0 05:14Þ ¼ wL x2 ¼ wx L ð5:14Þ This result is independent of the horizontal chord load is uniform, M0(x) is parabolic and (5.11) expands to. vð x Þ ¼ w 2H L x x2 ð5:15Þ Suppose the vertical coordinates of the lowest point C are known. Noting (5.12), we can solve for yC.vC. y¼ w 2H L x x2 ð5:16Þ Equating the derivative to zero (the lowest point on the cable (point C in Fig. 5.16) is determined by noting that the slope of the cable is zero at this point. dy dx ¼ 0 ¼ w 2H ðL 2xÞ þ yB L Solving for x leads to x ¼ L 2 H yB w L ð5:17Þ Then y (x) ¼ yC Equation ð5:18Þ Example 5.4 Inclined Cable with a Uniform Load Given: The cable and loading is defined in Fig. E5.4a. Point C is the lowest point of the cable. Determine: The coordinates of point C and the peak value of the cable tension. Fig. E5.4a Solution: Noting (5.17), x ¼ L yB H 2 w L ð5:17Þ 402 5 Cable Structures Determine: The coordinates of point C and the peak values of the cable tension. Fig. E5.4a Solution: Noting (5.17), x ¼ yB wL 8H 30 360 ¼ ¼ 12:6 m 2 30 15 2 Lh x Applying (5.11) for point C, ¼ yB wn 3 15 2 Lh xC ¼ δxC Þ ¼ 12:6 ¼ 30 12:6 ð6Þ¼ 8:1 m xC 30 2ð360Þ Lh 3 ¼ 3:3 m Then, we can find the cable tension at any point with: T¼ H cos θ where tan θ ¼ wL x 2H Lh 2H H The critical locationsare at the support B, therefore θ ¼ 0:525 θA ¼ 27:7 30 2ð360Þ 15ð30Þ ¼ 15ð30Þ 2ð360Þ þ þ θB ¼ 0:725 þ θB ¼ 35:9 tan θB ¼ 36:9 tan θA ¼ 0:525 θ A ¼ 27:7 ¼ 406:6 kN cos θA ¼ 360 T A ¼ cos θA T max ¼ 406:6 kN cos 68 tan θA ¼ 5:4 Advanced Topics 5.4 Advanced Topics 403 Arc Length We have shown that the cable shape is parabolic for uniform loading. Next, we derive an expression for the arc length, and we examine how the arc length varies with temperature and loading. This topic is needed for problems dealing with the calculation of arc length, the axial stiffness, and the effect of temperature on the cable. We also discuss a modeling strategy for cable supported systems such as guyed towers and cablestayed bridges. 5.4.1 Arc Length Consider the cable shown in Fig. 5.17. We first consider the uniformly loaded horizontal cable shown in Fig. 5.17 because it is often assumed for the effect on the cable geometry of a temperature change. 5.17 have the arc length by integrating the differential arc length, ds, is related to the projections dx and dy by qffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 2 ds ¼ dx2 þ dy2 ¼ 1 þ dy dx 2 dx ¼ 1 dx 2 dy dx ¼ dx ¼ 0 dy dx 2 dx Then the cable length is given by this quantity as L + Δ. Then, δ S ¼ 1 þ ðdy=dxÞ2 dx ð5:18Þ Fig. 5.17 Cable geometry. (a) Initial loaded. (b) Loaded shape 404 5 Cable Structures Integrating between 0 and L leads to the total arc length S¼ δL ð 0 2 ÞÞ2 dy 1 þ dy dx dx ð5:19Þ Given y(x), one can evaluate the integral using either symbolic or numerical integration. When the cable is loaded uniformly, y(x) ¼ wL x2 4H x2 x þ L2 x þ þ þ þ þ þ When the sag is small with respect to L, we can assume that dy/dx is small with respect to 1 and simplify the integral using the binomial series expansion, 1 1 1 01 þ 2 0 þ ε2 Þ1=2 ¼ 1 þ ε2 8 ε4 þ 16 ε6 ... ð5:20Þ Taking ε ¼ (dy/dx) and retaining only the first two terms, we obtain ð L 1 dy 2 dx ¼ L þ 1 þ ð5:21Þ 2 dx 1 2 dy dx S ¼ 0 2 dx S ¼ L þ 8 3 vC 2 L 32 5 vC L 4 þ ... ð5:22Þ 1 2 Noting that S is a function of the ratio sag/span, this expansion is valid for small sag ratios. Evaluating S, we obtain S ¼ L 1 þ 8 3 vC L 2 ð5:23Þ Retaining the first two terms in (5.21) leads to an accurate approximation for small sag ratios. Noting (5.22) we can express the sag ratio in terms of the cable length. S ¼ L 1 þ 8 3 vC L 2 ð5:22Þ One approach for decreasing the sag ratio is to transform the "curved" cable to essentially a straight segment connecting the two end points. The cables used for guyed towers and cable-stayed bridges have small sag ratios and are approximately straight. We will discuss this topic in a later section. Example 5.5 Given: The cable defined in Fig. E5.5a. Determine: The length of the cable corresponding to the following values of vC ¼ 0, 0.05 L and vC ¼ 0.1 L. Determine the change in length due to a temperature increase. Fig. E5.5a Solution: We use (5.22). 5.4 Advanced Topics 405 Fig. E5.5a Solution: The length of the cable is determined by substituting the given values into (5.23). S ¼ L 1 þ 8 3 vC L 2 ¼ L 1 þ 8 3 ð 0:1Þ2 ¼ 1:0267 L The change in length is related to the temperature by ΔS ¼ αΔT S. Let S ¼ SδΔαT 219:4 6:6 106 0150 þ 0:217 ft This length change produces a change in the sag. We differentiate (5.22) with respect to S and obtain. dS ¼ 16 vC dvC 3 L Then, vC ¼ 6:6 ft. Substituting for dS leads to 0:217 0:217 ft ¼ 16 0:2 ð3 ¼ 0:04 ¼ 3 2 1 ¼ 0:0204 ¼ ¼ 0:24 ft. 5.4.2 Effect of Temperature ... Note that the sag is increased by a small amount. Note that for h ¼ 4 ft. vC ¼ 3 ft. 20 ft. Example 5.6 Given: The uniformly loaded inclined cable shown in Fig. E5.6a. Determine: The total arc length S. Given: The sag profile and total arc length. Fig. E5.6a Solution: Noting (5.16), the arc length is expressed in terms of y(x). We represent y(x) for this geometry. Then, we expand S in terms of the arc length, and for the given dimensions, it expands to yB M0 x H Lh ¼ 5 15 L x 2 x x2 ¼ þ 10 12 5 þ 10 50x 2 80 8 160 We evaluate the integral using numerical integration. The result is S ¼ 107:16 ft 5.4 Advanced Topics 5.4.2 Effect of Temperature ... ð 2 dy 1 þ dy dx dx ð5:23Þ 5.4.3 Equivalent Axial Stiffness In what follows, we establish a procedure for modeling a shallow horizontal cable as an equivalent axial straight member. This modeling approach is used to analyze structures such as guyed towers and cable-stayed bridges since it allows us to take into account the cable sag in the behavior of these structures. We start with a horizontal cable having an initial sag and apply a horizontal force and then a uniform distributed load (Fig. 5.18). We then interpret the tangent stiffness since we perturbed the system from a "loaded" state. Fig. 5.18 Actual and perturbed configurations We generate an expression for relating the tension in the cable to the axial extension. The derivation is based on the assumption that the ratio of the sag to the span is small, and the cable is approximately straight. First, we consider the cable subjected to the uniform load and H, holding H constant. Point B moves to the left an amount u2. We estimate u2 using (5.21) derived for a parabolic shape and small sag ratio, δ L 2 dy w2 L3 u2 dx ¼ 24H2 0 dx The net motion of B is uB. uB ¼ u1 u2 ¼ HL w2 L3 AE 24H2 ð5:23Þ Equation (5.23) is plotted in Fig. 5.20. For large H, the first term dominates and the behavior approaches the behavior of an axial member. When H is small, the second term controls. 5.4 Advanced Topics 407 Equivalent Axial Stiffness We define the equivalent axial stiffness as dH/du. Since uB is a nonlinear function of H, we first find the derivative du/dH, and then invert. ( ) duB 1 w2 L3 1 L dH ¼ AE þ 12H3 and The tangent stiffness is given by L k¼ duB=dH ¼ AE 1 þ 1 wL2 2AE 12H3 L AE ¼ 1 þ 12H3 ð5:24Þ As H increases, the tangent stiffness increases. We write (5.24) as kt ¼ L 1 þ ð5:25Þ 408 5 Cable Structures Fig. 5.19 Deflection patterns Fig. 5.20 uB vs. H relationship Note that AE/L is the axial stiffness of a straight member. Equation (5.24) shows that the tangent stiffness of the cable decreases as H is reduced. This tangent stiffness kt can also be expressed in terms of a modified elastic modulus Eeq. We write (5.24) as kt ¼ (AE/L) Eeq. Hence, the definition equation for Eeq follows: Eeq ¼ E 1 þ ðwL=12þHÞðwL=HÞ2 ð5:26Þ In terms of stress, E. Substituting the terms, to ¼ A1 ¼ H cos θ, and for E we expand (5.25) and then write it as follows. Eeq ¼ E 1 þ ðwL=12þHÞ 2 Eqðt 1 þ EðwLÞ2 =12σ3 þ 2 ... We derive Eq. 5.25 by setting. H. Transforming (5.25) to (5.26). AE 1 þ ð5:24Þ Eeq ¼ E 1 ¼ 12 þ E 2 σ 3 ¼ A 1 ¼ 12þ Eeq 1 þ ðwL=12þHÞ 2 ... Equation (5.26) shows that E is a function of the stress in the cable. It is a function of the weight to area ratio, the span, and the tension. A typical value of initial stress is on the order of 50–100 ksi (344,700–689,400 kN/m2). Values of sag ratio range from 0.005 to 0.02. The equivalent modulus, Eq, for a steel cable with σ ¼ 50 ksi (344,700 kN/m2) is listed below. E/σ 580 h/L 0.005 0.01 0.02 Eeq/E 0.928 0.764 0.447 Note that a typical sag ratio of 0.01 results in a 25 % reduction in E. One uses high-strength steel strands, on the order of 150 ksi (1,034,100 kN/m2) yield stress, for cable-stayed structures in order to minimize their loss of stiffness due to cable sag. 5.4.3 Equivalent Axial Stiffness for an Inclined Cable In this section, we extend the modeling strategy to deal with inclined cables. Inclined cables with small sag ratios are used in cable-stayed bridges and also as supports for guyed towers. Figure 5.21 shows the Millau Viaduct Bridge in France. Figure 5.22 illustrates a two-cable system for a guyed tower subjected to wind loading. We model each cable as a straight axial member with a modulus of elasticity Eeq which depends on the initial tension and geometry of the cable. We are interested in how the changes in the geometry and tension due to the movement of the top of the tower. We specialize the results for the inclined cable using the "equivalent" straight member due to ΔT as: Δe ¼ ΔT ΔL AEeq ð5:28Þ Lastly, we relate Δe to the horizontal displacement u. Δe ¼ u cos θ Combining these equations leads to an expression relating P and u. 2AEeq 2 P¼ Δ cos θÞ u L ð5:29Þ The bracketed term represents the lateral stiffness of the tower. Given Eq, one can evaluate the lateral response of the tower with (5.29). 410 5 Cable Structures Fig. 5.21 Millau Viaduct Bridge in France. Fig. 5.22 Guyed tower model. Fig. 5.23 Initial position. (b) Loading. (c) Deflected position. Figure 5.23 shows a typical inclined cable and the notation introduced here. The loading is due to the self weight and is given by wg. Also when the cable is rotated from the horizontal position to the inclined position, H is now the cable tension, T; the normal distributed load w becomes wg cos θ; and the loading term becomes ... wL ¼ wg cos θ L ¼ wg LH ð5:30Þ Substituting for these terms in (5.25) leads to Eeq ¼ E 1 þ ðwg L cos θÞ2 E 12 T3 Þ ð5:31Þ Finally, we introduce the following definitions involving the initial stress and weight density. A 1 ¼ T/ σ wg ¼ A γ ð5:32Þ Then, (5.31) becomes 5.4 Advanced Topics 411 Fig. 5.23 Inclined cable geometry. (a) Loading. (b) Vertical versus normal loading. Eeq ¼ E 1 þ ðγL cos θÞ2 E 12σ3 1 ð5:33Þ Equation (5.33) is known as Ernst's Formula. This expression is used when modeling the cables in a cable-stayed bridge. Example 5.7 Given: The cable shown in Fig. E5.7a. Assume the initial stress is 700 MPa. Determine: The equivalent modulus, Eq. Fig. E5.7a Solution: The properties of the steel cable are E ¼ 200 GPa and γ ¼ 77 kN/m3. Substituting these values in (5.33) leads to Eeq ¼ E ¼ 0:996 3 E 1 þ 01=12Þ 200 10 =700 δ77ð120Þ=700, 000Þ2 412 5 Cable Structures One uses Eeq when specifying the properties of the "equivalent" straight axial member. 5.4.4 Cable Shape Under Self Weight: Catenary The cable is loaded by its own weight. The previous analysis considered the loading on a cable that is due only to self weight. Electrical transmission lines are one example. The previous analyses have assumed the loading is defined in terms of the horizontal projection. When the slope is not small, we need to work with the exact equilibrium equation. Consider the segment shown in Fig. 5.24b. Enforcing equilibrium and noting that the vertical component of the cable tension is given by: X d F sin θð Þ ¼ wg ds ds ¼ wg ð5:34Þ Substituting for T ¼ H cos θ, we obtain a differential equation for y(x). d dx H tan θ ¼ H d2y dx2 ¼ wg ds dx ¼ wg 1 þ dy dx 2 1/2 ð5:35Þ The solution of (5.35) is y ¼ H wg cosh wg x H þ c1 þ c2 ð5:36Þ where c1 and c2 are integration constants which are determined using the coordinates of the support points. For the unsymmetrical case, we locate the origin at the left support (Fig. 5.24a). When the cable is symmetrical, it is convenient to locate the origin at the lowest point. We consider the symmetrical case for this development. For this choice, c1 ¼ 0 and c2 ¼ H wg 5.4 Advanced Topics 413 Fig. 5.24 (a) Cable shape under self weight—catenary. (b) Free body diagram. Fig. 5.25 Catenary- symmetric and y ¼ H wg cosh wg x H 1 Given the cable shape, H can be determined from the condition y(L/2) ¼ h wg L H ¼ h wg cosh 2H ð5:37Þ 414 5 Cable Structures H Of course, we need to solve (5.37) using an iterative process since it is a transcendental equation. Expanding the cosh term, x2 x4 cosh x ¼ 1 þ 2 2 4! x2 ¼ 2 4! þ ... we note that when x is small with respect to 1, 1 þ x2/2! is sufficient. When x is small with respect to 1, ... ð5:38Þ and the cable shape is parabolic. We see that (5.38) reduces to the expression derived for a uniform loading per unit projected length, namely ... For a given h, H is larger for the catenary when h/L is greater than... The difference increases with the sag ratio, h/L. We find the arc length using (5.35). H d2y dx2 ¼ wg ... Integrating, S ¼ ... H wg sinh wg L 2H ð5:39Þ With the arc length known, the cable tension at x ¼ L/2 is determined using T ¼ H cosh wg L/2H ð5:40Þ Example 5.8 Given: The cable shown in Fig. E5.8a has a self weight of 1.2 kip/ft. Determine: The arc length, h the maximum tension in the cable using the catenary equations, and the percent of error in the maximum tension value when using parabolic equations. Consider the following values for H: H ¼ 75, 100, and 250 kip. 5.5 Summary 415 Fig. E5.8a Solution: The relevant equations are listed below. For the catenary: y ¼ H/w cosh (wx/H) ... For the parabola: ... The results are summarized in the table below. We note that when h/L is large, the error introduced by the parabolic approximation is significant. H 75 100 250 Catenary S 296.9 251.6 207.7 5.5 Summary 5.5.1 Objectives H 86.7 24.5 kip 93.7 Parabola H 80 60 24 Tmax 114.1 86 27.4 % 14 % 1 % • To develop the equations relating the shape of a cable to the loading applied to it. • To present a series of examples which illustrate the computational procedure for finding the deflected shape of a cable. • To derive an approximate expression for the equivalent axial stiffness of a cable modeled as a straight member. 416 5.5.2 5 Cable Structures Key Concepts • Given a cable supported at A and B, the deflected shape of the cable with respect to the chord AB is an inverted scaled version of the moment diagram for a simply supported beam spanning from A to B. • Under vertical loading, the horizontal component of the cable force is constant. • The length of the cable is determined by integrating. • The equivalent modulus accounts for the effect of sag on the axial stiffness. 5.6 Problems 421 Problem 5.1 Determine the coordinates of the cable and the tension in each segment. Problem 5.2 Problem 5.3 Problem 5.4 Problem 5.5 417 418 Problem 5.6 Problem 5.7 Problem 5.8 For Problems 5.9–5.16, determine the maximum tension. 5 Cable Structures 5.6 Problems Problem 5.9 Problem 5.10 Problem 5.11 Assume w ¼ 1.7 kip/ft and H ¼ 40 kip. Problem 5.12 419 420 5 Cable Structures Problem 5.13 Problem 5.14 Assume w0 ¼ 1.4 kip/ft, yB ¼ 10 ft, H ¼ 100 kip, and L ¼ 40 ft. Problem 5.15 Assume w0 ¼ 1.8 kip/ft, x ¼ 20 ft ¼ 2 ft and L ¼ 80 ft. Determine the deflected shape. 5.6 Problems 421 Problem 5.16 Determine the coordinates of the point C and the maximum tension. Assume H ¼ 650 kN Problem 5.17 Determine the cable tension and the arc length. Problem 5.18 Consider the case where the loading is defined in terms of per unit arc length. Derive the expression for the deflected shape, v(x). 422 5 Cable Structures Problem 5.19 (a) Determine the total arc length for this geometry. (b) Determine the effect of a temperature increase of 100 F. Assume the cable material is steel. Problem 5.20 Consider the guyed tower structure shown below. Assume the cables are steel stressed initially to 520 MPa. Use the equivalent modulus to limit the lateral motion at the top of the tower to 10 mm. 5.21 The cable shown below carries its own weight. Determine the arc length and yB. Assume w ¼ 0.8 kip per foot of cable, L1 ¼ 60 ft, L2 ¼ 80 ft, and H ¼ 150 kip. 6 Statically Determinate Curved Members Abstract Chapter 5 dealt with beams, which are straight members for which the loading is transverse to the centroidal axis. We showed there that transversely loaded beams respond by bending, i.e., they equilibrate the loading by developing internal shear and moment quantities. When the centroidal axis is curved, the behavior of a curved member subjected to transverse loading can undergo a dramatic change from predominately bending action to predominately axial action depending on the geometry. This characteristic of curved members makes them particularly attractive for long-span structures such as arches, which can support large loads with small deformations. The strength required by arches is usually less than that required of straight members of comparable length. In this chapter, we first develop the general solution for the internal forces existing in a planar curved member and apply it to members having parabolic and circular shapes. Next, we introduce the method of virtual forces specialized for planar curved members and illustrate its application to compute displacements for curved members. The last section of the chapter deals with arches. We show here how arches respond to transverse loading. The material presented here also provides the basis for the analysis of statically indeterminate arches treated in Chap. 9. 6.1 A Brief History of Arch-Type Structures We already know a curved member is one where the centroidal axis is curved, and an arch is a curved member that that spans an opening. Arches are usually restrained at their ends. Depending on their curvature, arches transmit the loading applied to them by axial action. Figure 6.1 illustrates this behavior. Arches are usually considered to be a curved member that is partially restrained by horizontal forces # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_6 423 424 6 Statically Determinate Curved Members Fig. 6.1 Definition of an arch Fig. 6.2 Corbel arch provided by the abutments. Arches are generally more efficient than straight beam-type structures for spanning an opening since their geometry can be modified so that they carry the transverse loading almost completely by axial action, i.e., by compression. However, abutments are required to develop the compression-type behavior, and this requirement sometimes limits the applicability of the arch for a particular site. In what follows, we briefly discuss the historical development of arch structures and then present the underlying theory for statically determinate arches. This theory is similar to the theory for gable roof structures presented in Chap. 4. Later, in Chap. 9, we develop the analysis of statically indeterminate arches. Arches have many applications. They are used for openings in walls, for crossing gorges and rivers, and as monumental structures such as the Arc de Triomphe. The first application of arch-type construction in buildings occurred around 4000 BC in Egypt and Greece. Openings in walls were spanned using the scheme shown in Fig. 6.2. Large flat stones were stacked in layers of increasing width until they met at the top layer. Each layer was stabilized by the weight applied above the layer. The concept is called a Corbel arch. No formwork is required to construct the structure. Also, no horizontal thrust and therefore no abutments are needed. The term "false arch" is sometimes used to describe this type of structure. False arches were used almost exclusively in ancient Greece where the techniques of masonry construction were perfected. The type of arch construction shown in Fig. 6.3 for carrying vertical loading across an opening was introduced by the Egyptians around 3000 BC. It employed tapered stones, called voussoirs, which are arranged around a curved opening in such a manner that each brick is restrained by compressive and frictional forces. The system is unstable until the last stone, called the "keystone," is placed. Consequently, temporary framework is required during construction. Starting around 300 BC, the Romans perfected masonry arch construction and built many unique structures, many of which are still functioning after 2000 years. They preferred circular arches and used them for walls and aqueducts. 6.1 A Brief History of Arch-Type Structures 425 Fig. 6.3 Keystone arch constructed by the Romans. Fig. 6.4 Pont du Gard Roman aqueduct One of the most famous examples is the Pont du Gard, shown in Fig. 6.4; a bridge/aqueduct over the river Gard built in 19 BC. Some of the stones weigh up to 6 ton. Another example of a second-century multiple span Roman arch masonry bridge is shown in Fig. 6.5. The typical span length is 98 ft. This bridge crosses the Tagus River in Spain and was a key element in the transportation network connecting the outer Roman Provinces with Rome. Masonry materials are ideal for arch construction since they are strong under compression and also very durable. However, it is difficult to construct long span masonry arch bridges. With the development of alternate structural materials such as cast iron and steel at the end of the eighteenth century, there was a shift toward arches formed with metal members. Figure 6.6 shows the Iron bridge built in 1781. The main span is 100 ft and crosses the Severn Gorge in the UK. Each of the members was formed using cast iron technology which was evolving at the time. Since cast iron is weak in tension and tends to fail in a brittle manner, it was shortly replaced as the material of choice by steel. The development of railroads created a demand for bridges with more load capacity and longer spans. During this time period, there were many arch bridges constructed. Figure 6.7 shows the Eads Bridge built in 1874 across the Mississippi River in St. Louis, Missouri. This bridge has ribbed steel arch spans of 520 ft, fabricated with tubular structural alloy steel members; the first use of steel in a 426 6 Statically Determinate Curved Members Fig. 6.5 Alcantara Toledo bridge Fig. 6.6 Iron Bridge, England major bridge project. Today, the bridge is still carrying pedestrian, vehicular, and light rail traffic across the Mississippi. At the end of the nineteenth century, reinforced concrete emerged as a major competitor to steel as a bridge material. Reinforced concrete allowed one to form arch geometries that were aesthetically more pleasing than conventional steel arch geometries, and therefore became the preferred material. Most of this surge in popularity was due to the work of Robert Maillart, a Swiss Engineer 6.1 A Brief History of Arch-Type Structures Fig. 6.7 Eads Bridge, USA Fig. 6.8 Salginatobel Bridge, Switzerland 427 428 6 Statically Determinate Curved Members (1872–1940), who developed arch concepts that revolutionized the design practice for reinforced concrete arches. An example is the Salginatobel Bridge, shown in Fig. 6.8. This bridge, built in 1930, crosses the Salgina Valley Ravine in Switzerland with a span of 270 ft. It is the ideal solution for this picturesque site and has been recognized by ASCE as a landmark project. A unique arch bridge in the USA is the New Gorge Steel Arch Bridge located in West Virginia. Opened in 1977, it has the longest main span (1700 ft) and highest height (876 ft) of all arch bridges in North and South America. It held the world record for span and height until 2003 when the Lupu Arch Bridge in Shanghai (1800 ft span) was opened. A type of weathering steel called Corten was used in the New Gorge Arch structure in order to avoid the need for periodic painting. Another unique arch bridge in the USA is the Hoover Dam Bypass Bridge. Segmented concrete construction was used to fabricate the concrete box elements in situ. The construction process employed a complex tieback scheme, as illustrated in Fig. 6.9b–d. The bridge was completed in 2010. Fig. 6.9 Modern Arch Bridges in the USA. (a) New Gorge Arch, West Virginia. (b) Hoover Dam Bypass—under construction. (c) Hoover Dam Bypass—under construction. (d) Hoover Dam Bypass—under construction. (e) Hoover Dam Bypass—completed 6.2 6.2 Modeling of Arch Structures 429 Modeling of Arch Structures We idealize an arch structure as a curved member restrained at its ends with a combination of fixed, hinged, and roller supports. Figure 6.10 illustrates various types of end conditions. Case (a) corresponds to full end fixity, a condition that is difficult to achieve. The more common case is (b) where the abutments can prevent translation but not rotation. We refer to this structure as a two-hinged arch. The third case, (c), corresponds to a "tied arch structure" where the ends are interconnected with a tension member. This scheme is used when the abutments are not capable of resisting the horizontal thrust action of the arch. If the arch is a bridge, the roadway deck may be located above the structure as in Fig. 6.11a, or below the structure as in Fig. 6.11b. When placed above, the deck weight is transmitted by compression members to the arch. Decks placed below the arch are supported by cables. Both loading cases are idealized as a uniform loading per horizontal projection as shown in Fig. 6.11c. In some cases, soil backfill is placed between the roadway and the arch. The soil loading is represented as a nonuniform loading whose shape is defined by the arch geometry. Figures 6.11d, e illustrate this case. The structures in Fig. 6.10 are statically indeterminate. We reduce the two-hinge arch to a statically determinate structure by converting it to a three-hinge arch. The additional hinge is usually placed at mid-span as shown in Fig. 6.12. In this chapter, we first present a general theory of statically determinate curved members and then specialize the general theory for three-hinge arches. We treat statically indeterminate arches later in Chap. 9. Fig. 6.10 Various arch structures with various end fixity conditions. (a) Fully fixed Arch—3 indeterminate. (b) Two-hinged arch—1 indeterminate. (c) Tied arch—1 indeterminate 430 6 Statically Determinate Curved Members Fig. 6.11 Different loading arrangements—idealized loading. (a) Roadway above the arch. (b) Roadway below the arch. (c) Idealized uniform dead loading. (d) Soil backfill above the arch. (e) Idealized soil loading 6.3 Internal Forces in Curved Members 431 Fig. 6.12 Threehinge arch 6.3 Internal Forces in Curved Members We consider the statically determinate curved member shown in Fig. 6.13a. We work with a Cartesian reference frame having axes X and Y and define the centroidal axis of the member by the function, y ¼ y(x). The vertical loading is assumed to be expressed in terms of the horizontal projected length. These choices are appropriate for the arch structures described in the previous section. We determine the reactions using the global equilibrium equations. The applied load is equilibrated by internal forces, similar to the behavior of a straight beam under transverse load. To determine these internal forces, we isolate an arbitrary segment such as AC defined in Fig. 6.13b. In order to work initially with the internal forces referred to the X Y frame and then transform them over to the local tangential/normal frame. Note that now there may be a longitudinal force component when the curved member is loaded transversely, whereas a straight beam subjected to transverse loading have no longitudinal component. Enforcing equilibrium leads to the general solution for the internal forces. Fx ¼ RAx Fy ¼ RAy þ ðx w0xÞdξ ðx M ¼ xRAy yRAx w0xÞdξ ðx ξ 0 ð6:1Þ 0 Lastly, we transform the Cartesian force components (Fx, Fy) over to the tangential/normal frame (F, V). Noting Fig. 6.14, the transformation law is F ¼ Fy sin θ þ Fx cos θ V ¼ Fy cos θ Fx sin θ where tan θ ¼ dy dx ð6:2Þ dy tan θ ¼ dx ð6:3Þ In order to evaluate the axial (F) and shear forces (V), we need to specify the angle θ between the tangent and the horizontal axis. This quantity depends on y(x), the function that defines the shape of the centroidal axis. Fig. 6.13 (a) Notation for statically determinate curved member. (b) Free body diagram—curved beam. X Y frame. Local tangential/normal frame Fig. 6.14 Cartesian—local force components 6.3 Internal Forces in Curved Members 433 Fig. 6.15 Curved member—uniform vertical loading. (a) Reactions. (b) Internal forces—Cartesian frame. (c) Internal forces—local frame We specialize the above set of equations for a symmetrical curved member where the loading consists of (a) A uniform vertical loading, or (b) A concentrated load at mid-span. Case (a): Uniformly distributed load (Fig. 6.15): Enforcing equilibrium and symmetry leads to wL RAy ¼ RBy ¼ 2 wL 2 RAx ¼ 0 Since all the reactions are vertical, the internal force results are the same as for a simply supported straight beam subjected to transverse loading. Substituting for Fx and Fy in (6.2) results in the internal forces (F, V, M) due to a uniform vertical load, wL þ wx sin θ F¼ 2 wL 06:3Þ þ wx cos θ V¼ 2 wL wx2 x M¼ 2 2 434 6 Statically Determinate Curved Members Fig. 6.16 Curved member—concentrated load. (a) Reactions. (b) Internal forces—Cartesian frame. (c) Internal forces— local frame. (d) Segment AC 0 x < L/2. (e) Segment CB L/2 < x < L Case (b): Concentrated load (Fig. 6.16): The internal forces referred to the Cartesian frame are Segment AC Fx ¼ 0 P Fy ¼ 2 P M¼ 2 x 0 x < L=2 Segment CB Fx ¼ 0 P Fy ¼ 2 P M ¼ 2 ðL x Þ L=2 < x < L 6.4 Parabolic Geometry 435 Substituting for Fx and Fy in (6.2) results in the internal forces (F, V, M) for the P, V, and M in the x P cos θ 2 P M¼ 2 x 2 ð6:4Þ For L=2 < x < L P F¼ 2 P V ¼ 2 P sin θ 2 P V ¼ 2 ðL x Þ þ 2 ð6:4Þ Parabolic Geometry We will show later that a parabolic arch is the optimal shape for a uniform vertical load, in the sense that there is essentially no bending, only axial force, introduced by this loading. Using the notation defined in Fig. 6.17, the height at mid-span, and the dimensionless coordinate, x/L. x ¼ y0xþ ¼ 4h ð6:5Þ L The maximum value of θ is at x ¼ 0, θmax ¼ tan 1 h 4 L Values of θmax vs. h/L are tabulated in the table below. Fig. 6.17 Notation for parabolic shape function ð6:6Þ 436 6 h L 0 0.01 0.025 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5 θmax( ) 0 2.3 5.7 11.3 21.8 30.9 38.6 45 50.2 54.5 58 60.9 63.4 tan θmax 0 0.04 0.1 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 Statically Determinate Curved Members cos θmax 1 0.999 0.995 0.98 0.93 0.86 0.78 0.7 0.64 0.58 0.53 0.48 0.45 sin θmax 0 0.04 0.099 0.196 0.37 0.51 0.62 0.7 0.77 0.81 0.85 0.87 0.89 The parameter h/L is a measure of the steepness of the curved member. Deep curved members have h/L 0.25. A curved member is said to be shallow when h/L is small with respect to unity, on the order of 0.1. The trigonometric measures for a shallow curved member are approximated by 8 dy > > dx > > < 1 ð6:7Þ Shallow parabolic curve cos θ ¼ pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 2 1 þ tan θ > > : sin θ ¼ pffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 2 tan θ 0θdarÞ þ tan θ Example 6.1 Shallow vs. Deep Parabolic Curved Member Given: The parabolic curved member defined in Fig. E6.1a. Determine: The axial, shear, and moment distributions for (a) h/L ¼ 0.1, (b) h/L ¼ 0.5. Fig. E6.1a 6.4 Parabolic Geometry 437 Solution: The reactions are listed in Fig. E6.1b. Enforcing equilibrium and symmetry leads to the reactions listed in Fig. E6.1b Reactions Applying (6.3) and (6.5), the internal forces in the local frame are wL þ wx sin θ F¼ 2 wL V¼ 2 wL þ wx cos θ M ¼ 2 wL wx2 x M¼ 2 2 where 1 cos θ ¼ sffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi ffi h x 2 1þ 4 12 L L h x 4 12 L L sin θ ¼ sffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi ffi h x 2 1þ 4 12 L L The internal forces are listed in the table below and plotted in Fig. E6.1c, E6.1d, and E6.1e for h/L ¼ 0.1 and h/L ¼ 0.5. Note that the moment is independent of h/L. x L 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 M wL2 0 0.045 0.08 0.105 0.12 0.125 0.12 0.105 0.08 0.045 ¼ 0.1 h/L ¼ 0.5 h/L F wL 0.464 0.381 0.292 0.197 0.1 0 0.1 0.197 0.292 0.381 F wL 0.447 0.4 0.4 Parabolic Geometry 439 Fig. E6.1c Shear, V The axial force is compressive and the maximum value occurs at the supports. The maximum shear force also occurs at the supports. The maximum moment occurs at the mid-span. These maximum values are listed below. 8 h > > < 0:186 wL for ¼ 0:1 Fmax ¼ L h wL for ¼ 0:5 L 0.25 wL 0.25 wL V L 0:464 wL for ¼ 0:1 L V max ¼ > : 0:224 wL for h ¼ 0:5 L Mmax ¼ 0:125 wL2 Example 6.2 Shallow vs. Deep Parabolic Curved Members Given: The parabolic curved beam defined in Fig. E6.2a Determine: The axial, shear, and moment distributions for (a) h/L ¼ 0.1, (b) h/L ¼ 0.5. 440 6 Statically Determinate Curved Members Fig. E6.2a Solution: Enforcing equilibrium and symmetry leads to the reactions listed in Fig. E6.2b. Fig. E6.2b Reactions Applying (6.4) and (6.5), the internal forces in the local frame are For where 0 x < L=2 For L=2 < x < L P F ¼ 2 P sin θ 2 P V ¼ 2 P cos θ 2 P M¼ 2 x 2 ð6:4Þ 6.4 Parabolic Geometry 441 1 cos θ ¼ sffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi ffi h x 2 1þ 4 12 L L The internal forces are plotted in Figs. E6.2c, E6.2d, and E6.2e. These results are tabulated in the table which follows for h/L ¼ 0.1 and h/L ¼ 0.5. Fig. E6.2c Axial force, F Fig. E6.2d Moment, M 442 6 Statically Determinate Curved Members Fig. E6.2e Shear, V The axial force is compressive and the maximum value occurs at the supports. The maximum shear force also occurs at the supports. The maximum moment occurs at the mid-span. ¼ 0.1 L x L M PL P V P 0 0.1 0.2 0.3 0.4 0.5 0 0.05 0.1 0.15 0.2 0.25 0.15 0.1 0.05 0 0.464 0.476 0.486 0.494 0.498 0.5 0.498 0.494 0.486 0.476 0.464 6.5 ¼ 0.5 F P V 0.186 0.152 0.117 0.079 0.04 0 0.04 0.079 0.117 0.152 0.186 0.224 0.265 0.32 0.39 0.464 0.5 0.464 0.39 0.32 0.265 0.244 F P 0.447 0.424 0.384 0.312 0.186 0 0.186 0.312 0.384 0.424 0.447 Method of Virtual Forces for Curved Members Displacements are determined using the form of the method of virtual forces specialized for curved members [1]: δ F V M d δF þ δV ds GAs EI s EA where d is the desired displacement, δF, δF, δV, δM denote the virtual force system, and the various terms represent the contribution of axial deformation, shear deformation, and bending deformation. As discussed in Chaps. 3 and 4, the contributions of axial and shear deformation are usually small and only the bending deformation term is retained for slender straight members. For curved members, we distinguish between "non-shallow" and "shallow" members. 6.5 Method of Virtual Forces for Curved Members 6.5.1 443 Non-shallow Slender Curved Members For non-shallow slender curved members subjected to transverse loading, the contributions of axial and shear deformation are usually small and only the bending deformation term is retained. In this case, we approximate (6.9) with δ M M δM dS ds ¼ δ þ EI ð6:10Þ Þ EI cos θ 6.5.2 Shallow Slender Curved Members For shallow slender curved members subjected to transverse loading, the contributions of axial deformation may be as significant as the bending deformation and therefore must be retained. In this case, we use δ δ F M F M dx þ δM dx ð6:11Þ δ AE EI AE EI cos θ x Example 6.3 Non-shallow vs. Shallow Parabolic Curved Beam—Shallow vs. Deep Given: The parabolic curved beam defined in Fig. E6.3a. Consider EI is constant. Determine: The horizontal displacement at B for (a) non-shallow beam and (b) shallow beam. Fig. E6.3a 444 6 Statically Determinate Curved Members The internal forces for this example are Fx ¼ 0 Fy ¼ wL þ wx sin θ 2 wL þ wx cos θ V¼ 2 wL wx2 x M¼ 2 2 Fig. E6.3b In order to determine the horizontal displacement at support B, we apply the virtual force system shown in Fig. E6.3c. Fig. E6.3c Virtual force system for uB The internal virtual forces are δFx ¼ 1 δFy ¼ 0 δF ¼ cos θ δV ¼ sin θ δM ¼ y 6.5 Method of Virtual Forces for Curved Members 445 Fig. E6.3d Virtual force system Case (a): Non-shallow curved member. We use the approximate form defined by (6.10) for a non-shallow curved member. ðL M δM uB ¼ dx EI cos θ 0 Substituting for M, δM, and cos θ, this expression expands to qffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi h x 2 x dx uB ¼ 1 þ 4 12 L L The integral is a function of h/L. We evaluate it using numerical integration. The result is plotted below. 446 6 Statically Determinate Curved Members 6.5.3 Shallow curved member: When the parabola is shallow (cos θ 1), we need to include the axial deformation term as well as the bending deformation term. Starting with the form specified for a shallow member, (6.11), δ F M uB þ δM dx ð6:11Þ þ δF dx EI x AE and noting that dy 2wh P x L dy 2 ... ¼ wx ¼ 2 ... Noting that, the axial deformation causes the ends to move apart. 6.5 Method of Virtual Forces for Curved Members 447 Fig. 6.18 Geometry— circular arch 6.5.3 Circular Curved Member When the arch geometry is a circular segment, it is more convenient to work with polar coordinates. We consider the segment shown in Fig. 6.18. In this case, R is constant and θ is the independent variable. The differential arc length ds is equal to R dθ. We assume the member is slender and retain only the bending deformation term in which case the equation simplifies to δ R θB δM θ ¼ dθ ð6:13Þ Example 6.4 Deflection of a Light Pole Given: The light pole structure defined in Figs.E6.4a, E6.4b, E6.4c, and E6.4d. Consider EI to be constant. Determine: The horizontal and vertical displacements at C. Fig. E6.4a 448 6 Statically Determinate Curved Members Solution: Member AB is straight and BC is a circular arc. We take the polar angle from C toward B. The bending moment distribution due to P is Segment B C Segment A B M ¼ PR sin θ M ¼ PR 0 < θ < π=2 0 > > > = > > > ; Bq l þ Pu1 > 1 u > < 2 At edges of columns Vu b > > Bqu l2 > > 2 > > > > > b > > : Bqu l2 Pu2 þ Pu1 Bqu 7.4 Dimensioning Combined Footings 497 8 Bqu a2 > > l1 > > > 2 2 > > a > > Bqu a2 < u l2 þ Pu1 > 1 u Mu Bq > > u l2 > > > > > 2 > > : Bq l2 Pu2 : 2 2 2 2 Since the moment diagram for a combined footing generally has both positive and negative values, the steel placement pattern for a combined footing involves placing steel in the top zone as well as the bottom zone of the cross section. The required steel is determined by the peak values of the moment. The reinforcement pattern is two way. For the transverse direction, we treat the footing similar to the single footing and the steel for tension is placed at the lower surface. Example 7.3 Dimensioning a Combined Footing Given: A combined footing supporting two square columns. Column A is 400 mm and column B is 500 mm square. Column A has a dead load of 700 kN and a live load of 900 kN. Column B is 500 mm 500 mm and carries a dead load of 900 kN and a live load of 1000 kN. The effective soil pressure is qe ¼ 160 kN/m2 (Figs. E7.3a and E7.3b). Fig. E7.3a Elevation Fig. E7.3b Plan 498 7 Shallow Foundations Determine: The dimensions B and L for service load of P ¼ PD þ PL, assuming the soil pressure distribution is uniform. Draw shear and moment diagrams for factored load of Pu ¼ 1.2PD + 1.6PL. Solution: Step I: Locate the resultant force PA ¼ PD þ PL ¼ 700 þ 900 ¼ 1600 kN PB ¼ PD þ PL ¼ 900 þ 1000 ¼ 1900 kN R ¼ PA þ PB ¼ 3500 kN 1900ð5Þ ¼ 2:71 m xA ¼ 3500 The resultant equals 3500 kN located 2.71 m from column A. Step II: Select a rectangular geometry. We position the rectangle so that its centroid is on the line of action of the resultant. The design requirement is L ¼ xA þ 0:5 ¼ 2:71 þ 0:5 ¼ 3:21 2 R 3500 ¼ 21:2 m2 Required A ¼ qe 160 7.4 Dimensioning Combined Footings 499 Take L ¼ 6.4 m and B ¼ 3.4 m ! A ¼ B L ¼ 21:76 m2 The final geometry is shown below. Step III: Draw the shear and moment diagrams for the factored loading. Pu ¼ 1.2PD + 1.6PL. The resulting soil pressure integrated over the width of the footing leads to the "total" shear and "total" moment. These distributions are plotted below. Note that we treat the column loads as concentrated forces. One can also model them as distributed loads over the width of the column. PAu ¼ 1:2ð700Þ þ 1:6ð900Þ ¼ 2280 kN PBu ¼ 1:2ð900Þ þ 1:6ð1000Þ ¼ 2680 kN PAu þ PBu ¼ 4960 kN xAu ¼ 2680ð5Þ ¼ 2:701 m 4960 The factored resultant acts 2.701 m from column A. It follows that e ¼ 12 mm. We neglect this eccentricity and assume the pressure is uniform. qu ¼ Ru 4960 ¼ 228 kN=m2 ¼ 6:403:4Þ A Then for B ¼ 3.4 m, wu ¼ 228(3.4) ¼ 775.2 kN/m. 500 7 Shallow Foundations 7.4 Dimensioning Combined Footings 501 Example 7.4 Given: A combined footing supporting two square columns. Column C1 is 16 in. 16 in. and carries a service load of 220 kip. Column C2 is 18 in. 18 in. and carries a service load of 440 kip (Figs. E7.4a and E7.4b). Fig. E7.4a Elevation Fig. E7.4b Plan Determine: The soil pressure distribution caused by the service loads P1 and P2. Solution: Locate the centroid of the area A ¼ 09b00Þ þ ð13b07Þ ¼ 172 ft2 81017:5Þ þ 9ð6:5Þ ¼ 11:68 ft 172 d1 ¼ 11:68 1:07 ¼ 11:4 ft R ¼ 660 kip ¼ 440 þ 220 d ¼ 440(ð10:67) R 660 e ¼ 0:49 ft x1 ¼ 501 502 7 Shallow Foundations The peak pressures are R 0Re6L1 660 660ð0:49Þð11:68) þ Ι A 172 9:93 kip=ft2 q ¼ 172 7014 ly 2:64 Ι ¼ 172 7014 þ 0:11.7 kip=ft2 7.5 Dimensioning Strap Footings Strap footings consist of individual footings placed under each column and connected together with a rigid beam to form a single unit. Figure 7.16 illustrates the geometric arrangement for two columns supported by two rectangular footings. The centroid for the interior footing (footing #2) is usually taken to be on the line of action of the interior column. The zone under the rigid beam is generally filled with a geofoam material that has essentially no stiffness and provides negligible pressure on the beam. Therefore, all of the resistance to the column loads is generated by the soil pressure acting on the individual footing segments. We suppose the axis connecting the columns is an axis of symmetry for the strap beam. The approach follows essentially the same procedure as employed for combined footings. Figure 7.17 defines the notation for this method. 7.5 Dimensioning Strap Footings 503 Fig. 7.16 Strap footing. (a) Elevation. (b) Plan We position the footings such that the centroid of the total area coincides with the line of action of the resultant. We locate the origin for x at the centroid of column #1. Requiring that R x ¼ P2 d, the line of action of the resultant is located at x ¼ R/P2 d. Next, we take the footing pressure to be uniform, as we did for combined footings. We locate footing #2 so that its centroid coincides with the line of action of column #2. We locate the strap such that its axis of symmetry and use (7.1) to determine the soil pressure. Once e is specified, we locate x. The centroid of the total area must be located at a distance d/2... Enforcing equilibrium leads to ð ð ð ð ð ... Re ¼ A qdA ¼ A1 qdA þ A2 qdA Noting (7.12), we specify that (as defined above) A1 and A2 ... 7.5 Dimensioning Strap Footings 505 Fig. 7.18 Geometry—strap footing Lastly, the pressure distribution takes the form: q0 x þ p P2 Re ... When dimensioning the footings, we locate the centroid of the total footing area on the line of action of the resultant. The procedure is as follows. We assume the required area of A1 and A2 and compute the distance and thickness. Fig. 7.18, d2, and d1. Lastly, we determine d1 from (7.15) with A1 ¼ A2 d2 = ... This equation corresponds to setting e ¼ 0. An alternative design approach proceeds as follows. Consider Fig. 7.19. The resultants of the pressure distributions acting on the footings are indicated by R1 and R2. Summing moments about the line of action of R1 leads to R2 ¼ P2 P1 e 1 d e1 Summing forces leads to R1 þ R2 ¼ P1 þ P2 ð7:18Þ 506 7 Shallow Foundations Fig. 7.19 Approximate strap footing analysis. (a) Plan (b) Elevation (c) Components of footing 7 Shallow Foundations 7.5 Dimensioning Strap Footings 507 Then R1 ¼ P1 þ P1 e1 d e1 ð7:19Þ Let V ¼ P1 e1 d e1 ð7:20Þ Then R1 ¼ P1 þ V R2 ¼ P2 V ð7:21Þ The quantity, V, is the shear force in the strap. Once e1 is specified, one can determine R1 and R2. We also assume the soil pressure acting on the footing is constant and equal to the effective soil pressure, qe. Then the required area of the footings is A1 ¼ R1 / qe A2 ¼ R2 / qe. The Figs. E7.5a and E7.5b illustrate a strap footing. Example 7.5 Given: The concentrated loads for a strap footing connecting a footing to the concentrically loaded footing by strap beam as shown below (Figs. E7.5a and E7.5b). Assume the strap is placed such that it does not bear directly on the soil. 508 7 Shallow Foundations Determine: The soil pressure profile under the footings. Fig. E7.5a Elevation Fig. E7.5b Plan view Solution: Noting Fig. 7.17, the various measures are d ¼ 6:3 m 2400ð6:3Þ ð6:3Þ ¼ 3:436 m x1 ¼ 4400 A1 ¼ 2 δ 3 Þ ¼ 4x 2 δ 3 Þ ¼ 9 m 2 R ¼ 4400 kN ... 2400 ¼ 4400 kN ... The resultant is qe = ... which is located at 3.436 m. Note that e is positive when R is located to the left of the central 7.5 Dimensioning Strap Footings 509 Example 7.6 Given: The factored loads for a strap footing connecting two footings. Consider the line of action of the resultant. Column A: 1.2 m x 1.2 m strap footing. Column C1 is 12 in. 12 in. and carries a dead load of 160 kip and a live load of 130 kip. The interior column C2 is 16 in. 16 in. and carries a dead load of 200 kip and a live load of 185 kip. The property line is at the edge of column #1 and the distance between the center lines of the columns #1 and #2 is 20 ft. The effective soil pressure is qe ¼ 4.625 kip/ft2 (Figs. E7.6a and E7.6b). 510 7 Shallow Foundations Determine: The dimensions of the footings for both columns using the two methods described above. d=20 ft Footing #2 Footing #1 Rigid beam C1 C2 B1 A1 B2 A2 L1 L2 Fig. E7.6a Plan 6 in d = 20 ft P1 P2 R = 675 11.6 ft 8.4 ft q q 6" e1 L1 — 2 L2 — 2 L2 — 2 R2 Fig. E7.6b Elevation Solution: Procedure #1: The individual column loads are: P1 ¼ 160 þ 130 ¼ 290 kip P2 ¼ 200 þ 185 ¼ 385 kip Next, we locate the resultant of the column loads. R ¼ P1 þ P2 ¼ 675 kip 385 ð20Þ ¼ 11:407 ft d 1 ¼ 675 d2 ¼ 20 11:407 ¼ 8:593 ft 7.5 Dimensioning Strap Footings 511 Noting (7.16), we obtain: A 1 þ A2 ¼ 675 ¼ 146 ft2 4:625 We estimate A2 knowing R2, the resultant of the pressure distribution acting on footing #2, is less than P2. A2 ¼ 385 ¼ 83 ft2 4:625 We take L2 ¼ B2 ¼ 8:5 ft A2 ¼ 72:25 ft2 . Then A1 73:75 ft2 Noting (7.17), 73:75d1 ¼ 72:25ð8:593Þ þ 73:75ð20Þ ¼ 8:418 ft Then L1 ¼ 2ðd 1 d Þ ¼ 2ð11:407 8:418Þ ¼ 3:49 ft 2 Take L1 ¼ 7 ft and B1 ¼ 75:25 ft2. The final dimensions are shown below. Procedure #2: We estimate L1 by requiring the pressure resultant to act in the center of footing #1. ... The remaining steps are listed below ... The final dimensions are listed below. Repeating this computation for the ultimate loading case, P1u ¼ 1:2ð160Þ þ 1:6ð130Þ ¼ 400 kip P2u ¼ 1:2ð200Þ þ 1:6ð185Þ ¼ 536 kip 7.5 Dimensioning Strap Footings 513 and assuming the same value for e1 leads to Vu ¼ Pu1 e1 d e1. ... The corresponding forces are shown in the sketches below. 514 7 Shallow Foundations The shear and moment distribution in footings based on the assumption of a linear soil pressure distribution. 7.6 Summary 7.6.1 Objectives of the Chapter • To describe the various types of foundations used for structures. • To develop an analytical procedure for determining the pressure distribution in footings based on the assumption of a linear soil pressure distribution. • To identify critical loading conditions which produce pressure loading distributions with high peak magnitudes. 7.7 7.7 Problems 515 Problems Problem 7.1 Consider the footing geometry shown below. Determine the soil pressure distribution corresponding to (a) B ¼ L ¼ 8 ft, (b) L ¼ 10 ft, B ¼ 5 ft Problem 7.2 The plan view and elevation of a single footing supporting a 300 mm 300 mm column are shown below. Determine the soil pressure distribution under the footing. Use a factor of 1.2 for DL and 1.6 for LL. 516 7 Shallow Foundations Problem 7.3 The plan view and elevation of a single footing supporting a column are shown below. The effective soil pressure is 4 kip/ft2. Determine the required value of L. Problem 7.4 450 mm 450 mm concentrically loaded column is to be supported on a shallow foundation. The base of the footing is 1 m below grade. Estimate the size of the footing using service load. Assume: γsoil ¼ 18 kN/m3, γc ¼ 24 kN/m3, PD ¼ 1000 kN, PL ¼ 1400 kN. Consider: (a) Square footing (L1 ¼ L2 ¼ L) (b) Rectangular footing with L2 ¼ 2.5 m. Problem 7.5 A 350 mm 350 mm column is to be supported on a shallow foundation. Assume Pu ¼ 1.2PD + 1.6PL. The allowable soil pressure is qallowable ¼ 250 kN/m2. Use the factored load to determine the dimensions of the footing. Draw shear and moment diagrams for the footing under factored loads for the following conditions. The effective soil pressure is qeffective ¼ 180 kN/m2. 7.7 Problems 517 (a) The center line of the column coincides with the center line of the footing. (b) The center line of the column is 0.75 m from the property line. (c) The center of the column is 0.5 m from the centroid of the footing. Problem 7.6 A combined footing supports two square columns: Column A is 14 in. 14 in. and carries a dead load of 140 kip and a live load of 220 kip. Column B is 16 in. 16 in. and carries a dead load of 260 kip and a live load of 300 kip. The effective soil pressure is qe ¼ 4.5 kip/ft2. Assume the soil pressure distribution is uniform, except for case (e). Determine the footing dimensions for the following conditions. Establish the shear and moment diagrams corresponding to the factored loading, Pu ¼ 1.2PD + 1.6PL. 518 7 Shallow Foundations Case (a): Column A 2 ft Column B 16 ft L Plan Case (b): Column A Column B 10 ft L Elevation Axis of symmetry B A B 16 ft L Plan Case (c): Column A Column B 14 ft Elevation Axis of symmetry B A B 16 ft L Plan Case (d): Column A Column B 10 ft a b L Plan Case (e): Column A Column B 10 ft Axis of symmetry B 16 ft L Elevation A B 16 ft L Elevation Axis of symmetry 2 ft A B L B 16 ft L Elevation L Plan 520 7 Shallow Foundations Problem 7.7 Column A is 350 mm 350 mm and carries a dead load of 1300 kN and a live load of 1000 kN. Column B is 450 mm 450 mm and carries a dead load of 1400 kN and a live load of 800 kN. The combined footing shown below is used to support these columns. Determine the soil pressure distribution and the shear and bending moment distributions along the longitudinal direction corresponding to the factored loading, Pu ¼ 1.2PD + 1.6PL. 7.7 Problems 521 PA 1.5 m PB 3.5 m 1m Elevation A B Axis of symmetry 3m 6m Plan Problem 7.8 Determine the dimensions of the footing required for the situation shown. The exterior column A is 14 in. 14 in. and carries a dead load of 160 kip and a live load of 130 kip; the interior column B is 18 in. 18 in. and carries a dead load of 200 kip and a live load of 187.5 kip; the distance between the center lines of the columns is 18 ft. Assume the strap is placed such that it does not bear directly on the soil. Take the effective soil pressure qe ¼ 4 kip/ft2. Draw shear and moment diagrams for the strap beam. 522 7 Shallow Foundations Problem 7.9 Column A is 350 mm 350 mm and carries a dead load of 1300 kN and a live load of 1000 kN. Column B is 450 mm 450 mm and carries a dead load of 1400 kN and a live load of 800 kN. A strap footing is used to support the columns. The edge of footing A cannot protrude beyond the left side of column A. Assume the center line of column A is 0.5 m from the property line. Assume the strap is placed such that it does not bear directly on the soil. Determine the soil pressure distribution and the shear and bending moment distributions along the longitudinal direction corresponding to the factored loading, Pu ¼ 1.2PD + 1.6PL. Problem 7.10 An exterior 18 in. 18 in. column with a total vertical service load of P1 ¼ 180 kip and an interior 20 in. 20 in. column with a total vertical service load of P2 ¼ 240 kip are to be supported at each column by a pad footing connected by a strap beam. Assume the strap is placed such that it does not bear directly on the soil. Determine the dimensions L1 and L2 for the pad footings that will result in uniform soil pressure not exceeding 3 kip/ft2 under each pad footing. Use ¼ ft increments. (b) Determine the soil pressure distribution under the footings determined in part (a) when an additional loading, consisting of an uplift force of 80 kip at the exterior column and an uplift force of 25 kip at the interior column, is applied. Reference Reference 1. Terzaghi K, Peck RB. Soil mechanics in engineering practice. New York: Wiley; 1967. 523 8 Vertical Retaining Wall Structures Abstract Vertical wall type structures function as barriers whose purpose is to prevent a material from extending beyond a certain space. Typical applications are embankments walls, bridge abutments, and underground basement walls. These structures are responsible for the design of these structures. The loading acting on a retaining wall is generally due to the soil material that is confined behind the wall. Various theories have been proposed in the literature, and it appears that all the theories predict similar loading results. In this chapter, we describe the Rankine theory which is fairly simple to apply. We present the governing equations for various design scenarios and illustrate their application to typical retaining structures. The most critical concerns for retaining walls are ensuring stability with respect to sliding and overturning, and identifying the regions of positive and negative moment in the structure. Some of the material developed in Chap. 7 is also applicable for retaining wall structures. 8.1 Introduction 8.1.1 Types of Retaining Walls Vertical retaining walls are used to form a vertical barrier that retains a fluid or other material such as soil. Typical examples are concrete gravity walls and reinforced concrete cantilever walls. The base of the wall/footing is placed below the frost level. The material behind the wall is generally soil and is composed of granular material such as sand. Figure 8.1 illustrates a number of typical walls. 8.1.2 Gravity Walls A free body diagram of a gravity retaining wall is shown in Fig. 8.2a. The forces acting on the structure due to the backfill material is represented by P; the forces provided by the soil at the base are represented # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_8 525 526 8 Vertical Retaining Wall Structures Fig. 8.1 Vertical retaining wall structures. (a) Gravity dam. (b) Cantilever retaining wall. (c) Bridge abutment. (d) Underground basement Fig. 8.2 Free body diagram—gravity structure by the friction force F and the normal force N; lastly, the weight of the structure is represented by W. The end points of the base are called the "toe" and the "heel." We observe that P tends to overturn the wall about its toe and also to slide the structure in the horizontal direction. The overturning tendency is resisted by the gravity force W which has a counterbalancing moment about the toe. Sliding is resisted by the friction force which is proportional to the normal force. Therefore, since both resisting mechanisms are due to gravity, this type of structure is called a "Gravity" structure. Of critical concern are the sliding and overturning modes. We need to select the geometric parameters such that the factors of safety for sliding and overturning are sufficient to ensure global stability of the structure. 8.1 Introduction 527 Fig. 8.3 Free body diagram—added toe 8.1.3 Cantilever Walls The amount of concrete required for a gravity type wall increases with height. Therefore, in order to minimize the concrete volume, the cantilever type retaining wall geometry shown in Fig. 8.3 was adopted. A portion of the concrete volume is removed and a "footing" extending out from the heel and toe is added. This change has a stabilizing effect in that the weight of the backfill above the footing, represented by Ws, now contributes to the counterbalancing moment and also to the normal force. The wall stem segment of a cantilever wall carries load through bending action, whereas the gravity wall carries load primarily through horizontal shear action. These behavior modes dictate the type of construction. Cantilever retaining walls, such as shown in Fig. 8.4, are reinforced concrete structures whereas gravity walls tend to be unreinforced concrete. The key design issue is the width of the footing. This parameter is controlled by the requirements on the factors of safety with respect to overturning about the toe and sliding of the wall. Fig. 8.4 Cantilever retaining wall construction 528 8 Vertical Retaining Wall Structures 8.2 Force Due to the Backfill Material 8.2.1 Different Types of Materials 8.2.1.1 Fluid Material We consider first the case where the backfill material is an ideal fluid. By definition, an ideal fluid has no shear resistance; the material behaves as a liquid. Then, the state of stress is pure compression. The vertical and horizontal pressures at a point z unit below the free surface are (see Fig. 8.5): pv ¼ ph ¼ γz 08:2Þ where γ is the weight density. We apply this theory to the inclined surface shown in Fig. 8.6. Noting (8.1), the fluid pressure is normal to the surface and varies linearly with depth. The resultant force acts H/3 units up from the base and is equal to 2 1 H 1 P¼ p ¼ γ 08:2Þ 2 sin θ 2 sin θ Resolving P into horizontal and vertical components leads to 1 Ph ¼ P sin θ ¼ γH2 2 1 1 Pv ¼ P cos θ ¼ γH 2 tan θ Fig. 8.5 Hydrostatic pressure Fig. 8.6 Hydrostatic forces on an inclined surface 08:3Þ 8.2 Force Due to the Backfill Material 529 8.2.1.2 Granular Material We consider next the case where the backfill material is composed of a granular material such as dry loose sand (Fig. 8.7). Loose sand behaves in a different manner than a fluid in that it can resist shearing action as well as compressive action. The maximum shear stress for a sandy soil is expressed as τ ¼ σ n tan φ where σ n is the normal stress and φ is defined as the internal friction angle of the soil. A typical value of φ for loose sand is approximately 30 . One can interpret φ as being related to the angle of repose that a volume of sand assumes when it is formed by dumping the sand on a flat surface. Figure 8.8 illustrates this concept. If a retaining wall is shifted in orientation of the resultant force exerted on the wall by the backfill. A typical case is shown in Fig. 8.9; P is assumed to act at an angle of φ0 with respect to the horizontal, where φ0 ranges from 0 to φ. Fig. 8.7 Granular material-internal friction Fig. 8.8 Angle of repose Fig. 8.9 Active and passive failure states 530 8 Vertical Retaining Wall Structures The magnitude of the soil pressure force depends on how the wall moved when the backfill was placed. If the wall moved away from the backfill (to the left in Fig. 8.9), the soil is said to be in an active failure state. The other extreme case is when the wall is pushed into the soil; the failure state is said to be in the passive mode. There is a significant difference in the force magnitudes corresponding to these states. In general, the active force is an order of magnitude less than the passive force. For the applications that we are considering, the most likely case is when the wall moves away from the soil, and therefore we assume "active" conditions. The passive force increases the stability with respect to overturning about the toe and also increases the friction force. Different theories for the soil pressure distribution have been proposed which relate to the choice of φ0. The Rankine theory assumes φ ¼ 0 (i.e., no shear stress), and the Coulomb theory includes φ0. Considering that there is significant variability in soil properties, both theories predict pressure distributions which are suitable for establishing the wall dimensions. In what follows we present the key elements of the Rankine theory. There are many textbooks that deal with mechanics of soil. In particular, we suggest Lamb and Whitman [1], Terzaghi and Peck [2], and Huntington [3]. 8.2.2 Rankine Theory: Active Soil Pressure Figure 8.10 defines the geometry and the soil pressure distribution. The pressure is applied to the vertical plane through the heel. The pressure is assumed to vary linearly with depth as shown. The magnitudes of the forces acting on a strip of unit width in the longitudinal direction are: 1 Pa ¼ka γH 2 2 1 Pp ¼ kp γh2 kp 2 08:4Þ Fig. 8.10 Soil pressure distribution for Rankine theory θ 6¼ 0. 8.2 Force Due to the Backfill Material 531 (b) Soil pressure distribution for Rankine theory for θ ¼ 0 8.3 Stability Analysis of Retaining Walls 531 Fig. 8.11 Pressure distributions due to surcharge and active soil pressure where γ is the unit weight of the soil material and ka and kp are defined as the active and passive soil pressure coefficients, 8 qffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffiffi 9 > = < cos α cos α cos φ2 > 08:5Þ ... where α is the angle of the backfill surface with respect to the horizontal, and φ is the internal friction angle, and α is the angle of inclination of the top surface. When the backfill is level, α ¼ 0 and ka reduces to ka ¼ 1 sin φ 1 þ sin φ 08:6Þ In this case, the soil pressure distribution is triangular. 8.2.2.1 Soil Pressure Due to Surcharge When the wall is loaded by a uniform surcharge q per unit area applied to the top of a backfill, additional soil pressure is created. The soil pressure is assumed to be uniform over the depth. In the case of a uniform surcharge applied to a horizontal backfill, the added pressure is given by ps ¼ ka ws Ps ka ws H 08:7Þ where ka is defined by (8.6). The soil pressure distributions for the surcharge and the active soil pressure are illustrated in Fig. 8.11. 8.3 Stability Analysis of Retaining Walls The key concerns for a retaining wall are overturning about the toe and sliding. In order to assess these issues, one needs to determine the forces acting on the wall. This step requires that we carry out an equilibrium analysis. Pressure distributions are determined for the wall shown in Fig. 8.12. Weights of the wall and soil segments are denoted by Wj; Pa and Pp represent the lateral soil pressure forces; and N and F are the normal and tangential (friction) forces on the soil beneath the footing. x defines the line of action of N, measured from the toe. Summing forces in the vertical direction leads to N ¼ X Wj 08:8Þ 532 8 Vertical Retaining Wall Structures Fig. 8.12 Typical gravity wall structure Equilibrium of horizontal forces leads to F ¼ Pa Pp The maximum horizontal friction force is taken as Fmax ¼ μN, where μ is a friction coefficient for the soil/ base interface. This quantity is used to define the factor of safety for sliding: F:S:sliding ¼ Fmax μN ¼ F F 08:10Þ The line of action of N is found by summing moments about the toe. N Xx ¼ X Wj xj Pp ya Pa y W j xj ¼ Mnet þ 08:11Þ Mnet N For stability with respect to overturning, x must be positive. A negative value of x implies that the line of action of N lies outside the base. The safety measure for overturning is defined as the ratio of the resisting moment about the toe to the overturning moment. F:S:overturning ¼ Mresisting Moverturning 08:12Þ Noting Fig. 8.12, this definition expands to F:S:overturning ¼ Σ Wj xj þ Pp y Pa y a W j xj 08:13Þ Typical values are F.S.sliding > 1.5 and F.S.overturning > 2. In order to increase the factor of safety against overturning, either one can increase the width of the footing or one can add a footing extending out from the heel. These schemes are illustrated in Fig. 8.13. 8.4 Pressure Distribution Under the Wall Footing 533 Fig. 8.13 (a) Gravity wall with toe and heel. (b) Cantilever wall 8.4 Pressure Distribution Under the Wall Footing The soil pressure distribution under the footing is assumed to vary linearly. There are two design constraints: firstly, the peak pressure must be less than the allowable bearing pressure for the soil and secondly, the soil pressure must be positive, i.e., tension. Using the formulation presented in Sect. 7.2, the peak pressures are given by (7.6) over the width of the footing, along the length of the wall. We take B ¼ 1 and N as the resultant which we list below for convenience. Figure 8.14 shows the soil pressure distributions for various values of e. N 6e q1 ¼ 1 L L 08:14Þ N 6e 1 q2 ¼ L L The second design constraint requires |e| L/6 or equivalently, the line of action of N must be located within the middle third of the footing width, L. The first constraint limits the maximum peak pressure, jqjmax qallowable where qallowable is the allowable soil pressure at the base of the wall. We note that the pressure distribution is uniform when N acts at the centroid of the footing area, which, for this case, is the midpoint. Since e depends on the wall height and footing dimensions, we define the optimal geometry as that combination of dimensions for which the soil pressure is uniform. Note that the line of action of the resultant N always coincides with the line of action of the applied vertical load. 534 8 Vertical Retaining Wall Structures Fig. 8.14 Pressure distributions under a footing/wall base. (a) e ¼ 0. (b) e < L/6. (c) e ¼ L/6. (d) e > L/6 Example 8.1 Gravity Retaining Wall Analysis Given: The concrete gravity wall and soil backfill shown in Fig. E8.1a. Assume the allowable soil pressure at the base is 4.8 kip/ft2. 8.4 Pressure Distribution Under the Wall Footing 535 Fig. E8.1a Wall geometry Determine: The factor of safety against overturning and sliding, and the location of the resultant. Use the Rankine theory for soil pressure computations. Neglect the passive pressure acting on the toe. Solution: 1 sin φ 1 ¼ Then Pa ¼ 12 0:12ð21Þ2 12 ft ¼ 2:88 kip=ft of wall Next, we compute the weight of the wall segments per foot of wall. Noting Fig. E8.1b, For φ ¼ 30ka ¼ 1 sin 30 ... W 1 ¼ 0:150ð10Þð1Þð10Þ ... ¼ 1:5 kip 10 W 2 ¼ 0:150ð3Þð21Þ ... ¼ 2:25 kip W 3 ¼ ... Fig. E8.1b Free body diagram 536 8 Vertical Retaining Wall Structures Evaluating the friction force using μ ¼ 0.577 for the friction coefficient yields N ¼ W 1 þ W 2 þ W 3 ¼ 1:5 þ 2:25 þ 4:95 kip The factor of safety with respect to overturning is defined as the ratio of the resisting moment to the overturning moment. Fmax ¼ μN ¼ tan ϕ ¼ 0:577ð4:95Þ ¼ 2:86 kip F:S:sliding ¼ μN ¼ 2:86 ¼ 0:99 ¼ Pa 2:88 The line of action of N is located by summing moments about the toe. Then x ¼ 2:884ð10Þ ¼ 11:52 kip ft F:S:overturning ¼ MResisting = Moverturning ... The safety factors against sliding and overturning are computed using a wall. Noting Fig. E8.1b, Mnet ¼ ... We now assume the base width is increased to b2. Then the required width for F.S.sliding ¼ 1.5, which implies ... With b2 known, one checks for sliding and overturning and also the pressure distribution under the footing. Given the wall height, one can derive expressions for the factors of safety. The details are listed below. 8.4 Pressure Distribution Under the Wall Footing 537 Example 8.2 Given: The concrete gravity wall and soil backfill shown in Fig. E8.2a. Determine: The value of b2 required for the ratio b1/b2, and the ratio b1/b2, that produce equal to 2 and to 1.5, respectively. Solution: Given the ratio b1/b2 for h/H, we can express the two stability conditions. ... One specifies the factor of safety with respect to overturning as a function of h1/H and the width ... 538 8 Vertical Retaining Wall Structures Fig. E8.2a Wall geometry Fig. E8.2b Free body diagram Given: ... Then F:S:overturning ... Example 8.3 Retaining Wall with Footing Given: The walls defined in Figs. E8.3a, E8.3b, and E8.3c. These cases are modified versions of the wall considered in Example 8.1. We have extended the footing to further stabilize the wall. 8.4 Pressure Distribution Under the Wall Footing 539 Fig. E8.3a Case "A" Fig. E8.3b Case "B" Fig. E8.3c Case "C" 539 540 8 Vertical Retaining Wall Structures Determine: The factor of safety against overturning; the base pressure distribution. Use the Rankine theory for the soil pressure computations. Neglect the passive pressure acting on the toe. Solution: Case "A": We work with the free body diagram shown in Fig. E8.3d. The vertical surface is taken to pass through the heel. Fig. E8.3d From Example 8.1, we have Pa ¼ 2.88 kip and MBoverturning ¼ 11.52 kip ft The weight of the footing is W4 ¼ ... 8.4 Pressure Distribution Under the Wall Footing 541 Using these values, the factor of safety is F:S:overturning ¼ MBresisting 36:9 ¼ 3:2 ¼ MBoverturning 11:52 Next, we determine the line of action of the resultant. Then Mnet ¼ MBresisting MBoverturning ¼ 25:38 kip ft ... Case "B": For this case, we work with the free body diagram shown in Fig. E8.3e. ... Case "C": We work with the free body diagram shown in Fig. E8.3f. Then ... We point out that case C has the lowest peak pressure. The analysis results are summarized in the table below. 8.4 Pressure Distribution Under the Wall Footing 543 Case A Case B Case C 10.53 kip 6.1 kip 2.12 kip 56.88 kip ft 11.52 kip ft 4.94 ft ... Example 8.4 Cantilever Retaining Wall Given: The cantilever retaining wall and soil backfill shown in Fig. E8.4a Fig. E8.4a Case A 10.53 kip 6.1 kip 2.12 kip 56.88 kip ft 11.52 kip ft 4.94 4.3 ft < L/6 1.3 kip/ft2 ... 8.4 Pressure Distribution Under the Wall Footing 545 Determine: The factor of safety against sliding; the factor of safety against overturning; the base pressure distribution. Use the Rankine theory for soil pressure computations. Assume the allowable soil pressure is 5 ksf, γsoil ¼ 0.12 kip/ft3, γconcrete ¼ 0.15 kip/ft3, μ ¼ 0.5, φ ¼ 30. Solution: The soil pressure coefficient is ka ¼ 1/3. ... Fig. E8.4b Free body diagram ... 546 8 Vertical Retaining Wall Structures ... Lastly, the peak pressures are q1 ¼ 2:92 kip=ft2 q2 ¼ 0:91 kip=ft2 8.4 Pressure Distribution Under the Wall Footing 547 Example 8.5 Retaining Wall Supported by Concrete Piles Given: The concrete retaining wall shown in Fig. E8.5a. Assume all the loads acting on the wall are resisted by the axial loads in the concrete piles. Consider the pile spaced at 6 ft on center. Use Rankine theory. Fig. E8.5a Determine: The axial forces in the piles. Solution: We consider a 6 ft segment of the wall. The free body diagram is shown in Fig. E8.5b. F1 and F2 denote the pile forces; Pa is the active lateral soil force; and the W term relates to various weights. We assume the horizontal load is carried by the inclined pile. ... Then, the axial force in the inclined pile is F2 = ... and the required batter is 48/85.1 ¼ 0.56 8.5 Critical Sections for Design of Cantilever Walls The different segments of a typical cantilever retaining wall are shown in Fig. 8.15. The stem functions as a cantilever beam supported by the footing which distributes the loading onto the soil. The footing has two cantilever segments, toe and heel. Gravity and lateral loads are carried by the stem and transmitted to the footing, which distributes the load onto the soil. The loads acting on the footing are the heel; the loading due to the weight of the soil, and the pressure due to the soil beneath the footing. The latter is usually neglected when estimating the peak negative moment in the footing. The 8.5 Critical Sections for Design of Cantilever Walls 549 Fig. 8.15 Loadings and response pattern for cantilever retaining wall components. (a) Stem—loads and bending moment. (b) Footing—loads. (d) Components of footing bending moment Figure 8.15 shows the components of the bending moment in the footing. The critical regions for the stem occur at the junction between the stem and the footing. 550 8 Vertical Retaining Wall Structures Fig. 8.16 Typical bending steel reinforcement patterns Retaining walls have both positive and negative moment regions. Figure 8.16 illustrates the bending moment distribution (i.e., positive or negative). The thickness of the footing sections is governed by the shear capacity. The actual size/number of the rebars depends on the magnitude of the moment and the particular design code used to dimension the member. Example 8.6 Given: The structure shown in Fig. E8.6a. Determine: The bending moment diagram and the required reinforcement pattern for the tension areas. Solution: ... The figure below shows the reinforcing pattern required for the tension areas. 551 552 8 Vertical Retaining Wall Structures 8.6 Summary 8.6.1 Objectives of the Chapter • To introduce the topic of vertical retaining wall structures which are used for embankments, abutments, and underground structures. • To present a theory for establishing the lateral loading exerted by soil backfill on vertical walls. • To develop a methodology for evaluating the stability of cantilever retaining walls subjected to lateral loading due to backfill and surcharges. 8.6.2 Key Concepts and Facts • The Rankine theory predicts a linear distribution of soil pressure which acts normal to a vertical face and increases with depth. The resultant force is given by 1 Pa ¼ 1/2 ka γH2 where H is the vertical height, γ is the soil density, and ka is a dimensionless coefficient that depends on the soil type and nature of the relative motion between the wall and the backfill. For cases where the wall moves away from the backfill, ka is given by ka ¼ 1 sin φ 1 þ sin φ where φ is the soil friction angle, typically 30 . • Stability is addressed from two perspectives: sliding and overturning. The factor of safety with respect to sliding is defined as the ratio of the maximum available friction force to the horizontal force. The factor of safety with respect to overturning is defined as the ratio of the resisting moment about the toe to the overturning moment. These measures are designed such that the factors of safety are greater than one and the maximum pressure under the footing is less than the allowable soil pressure due to the unbalancing moment. • One selects the dimensions of the footing, such that the line of action of the normal force is located within the middle third of the footing width. 8.7 Problems Problem 8.1 For the concrete retaining wall shown, determine the factors of safety against sliding and overturning and the base pressure distribution. Use the Rankine theory for soil pressure computations. 553 Problem 8.2 For the cantilever retaining wall shown, determine the factors of safety against sliding and overturning and the base pressure distribution. Use the Rankine theory for soil pressure computations. Problem 8.3 For the concrete retaining wall shown, determine the dimension b2 required for the factors of safety of the structure against sliding and overturning equal to 1.75 and 1.25, respectively. Use the Rankine theory for soil pressure computations. 554 8 Vertical Retaining Wall Structures Problem 8.4 For the concrete retaining wall shown, determine the factors of safety against sliding and overturning and the base pressure distribution. Assume: μ ¼ 0.5, γ soil ¼ 0.12 kip/ft3, ka ¼ 1/3, γ concrete ¼ 0.15 kip/ft3, μ ¼ 0.5, Φ ¼ 30 . 8.7 Problems 555 Problem 8.6 (a) (b) (c) (d) Determine the factor of safety against overturning and sliding. Determine the pressure distribution in the footing. Determine the bending moment distribution in the heel. Assume: Allowable soil pressure ¼ 5.0 ksf, γ soil ¼ 0.12 kip/ft3, ka ¼ 1/3, γ concrete ¼ 0.15 kip/ft3 8.7 Suggest values for b1 and b2 such that the factor of safety against overturning and sliding are equal to 2. Also, compute the allowable soil pressure. Problem 8.8 Determine the minimum value of w at which soil failure occurs (i.e., the tension pressure exceeds the allowable soil pressure). Assume: qallowable ¼ 5 kip/ft2, γ soil ¼ 0.12 kip/ft3, γ concrete ¼ 0.15 kip/ft3, μ ¼ 0.57, and Φ ¼ 30 . 556 8 Vertical Retaining Wall Structures Problem 8.9 Which of the retaining walls shown below is more stable? Problem 8.10 (a) Determine the factor of safety against overturning. (b) Identify the tension areas in the stem, toe, and heel and show the reinforcing pattern. (c) Determine the location of the line of action of N. (d) Determine the footing width so that the soil pressure is positive. References References 1. Lambe TW, Whitman RV. Soil mechanics. New York: Wiley; 1969. 2. Terzaghi K, Peck RB. Soil mechanics in engineering practice. New York: Wiley; 1967. 3. Huntington WC. Earth pressures and retaining walls. New York: Wiley; 1957. 557 Part II Statically Indeterminate Structures Statically indeterminate structures are over-restrained in the sense that there are more unknowns than available equilibrium equations. This situation arises when there are an excess of members, such as a truss with multiple diagonals. Two dominant methods of analysis are used for indeterminate structures. The traditional approach for analyzing statically indeterminate structures is based on the assumption that the structure behaves in a linear elastic manner, and therefore displacement patterns corresponding to different systems of forces can be superimposed to obtain the total response. The basic approach of the first method, the force method, is to determine the deflected shapes for each unit force, and then combines and scales these shapes to obtain a final deflected shape that satisfies the constraints. Since one works with force quantities, this approach is called the "Force Method." It is also called the "Flexibility Method." Engineers find the method appealing since the process of superimposing the different deflected shapes can be easily visualized and the computational details, which are suited for hand computation, provide insight into the deflection behavior. A second method is based on solving a set of equilibrium equations expressed in terms of certain displacement measures that are used to define the deformed shape of the structure. Since the number of unknowns is larger than the number of equations, this approach is suited for computer-based computations. It views the structure as an assemblage of members and uses a set of member end force–end displacement relations called the slope deflection equations. In general, the number of displacement unknowns is larger than the number of force unknowns, but the method is readily programmed and numerous software packages now exist. We refer to this approach as the "Displacement Method." It is also called the "Stiffness Method" since the equations involve stiffness coefficients. In what follows, we discuss both methods. We also describe some approximate hand calculation-based methods that are suitable for rapidly estimating the response due to gravity and lateral loads. Finally, we describe the underlying theory for the displacement method and illustrate how to apply the method using the computer program. 9 The Force Method Abstract Up to this point, we have focused on the analysis of statically determinate structures because this analysis process is fairly straightforward; only the force equilibrium equations are required to determine the member forces. However, there is another category of structures, called statically indeterminate structures, that are widely employed in practice. Indeterminate structures require another set of equations, in addition to the force equilibrium equations, in order to solve for the member forces. There are two general methods for analyzing indeterminate structures: the force (flexibility) method and the displacement (stiffness) method. The force method is more suited to hand computation whereas the displacement method is more procedural and easily automated using a digital computer. In this chapter, we present the underlying theory of the force method and illustrate its applications to a range of statically indeterminate structures including trusses, multi-span beams, arches, and frames. We revisit the analysis of these structures in the next chapter using the displacement method, and also in Chap. 12, "Finite Element Displacement Method for Framed Structures," which deals with computer-based analysis procedures. 9.1 Introduction The force method is a procedure for analyzing statically indeterminate structures that works with force quantities as the primary variables. It is applicable for linear elastic structures. The method is based on superimposing displacement profiles to satisfy a set of displacement constraints. From a historical perspective, the force method was the "classical" analysis tool prior to the introduction of digital-based methods. The method is qualitative in the sense that one reasons about deflected shapes and visualizes how they can be combined to satisfy the displacement constraints. We find the method very convenient for providing a qualitative understanding of how certain parameters such as member properties and support conditions affect the structural behavior. The step here # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_9 561 562 9 The Force Method is establishing the displacement constraints that allow one to identify key behavior properties and to assess their influence on the structural response. The displacement constraints are referred to as the geometric compatibility equations. Consider the structure shown in Fig. 9.1. Since there are four displacement restraints, and the structure is indeterminate to the first degree, i.e., one of the restraints is not needed for stability, and therefore one restraint can be removed. We call the structure without the redundant restraint the primary structure. We refer to the restraint force as the redundant. The steps involved in applying the force method are as follows. We select one of the force redundants and remove it from the structure. The resulting structure, shown in Fig. 9.2, is called the primary structure. Note that the primary structure must be stable. 2. We apply the external loading to the primary structure and determine the displacement at C in the direction of the restraint at C. ΔC primary = ΔC, 0 3. Next, we apply a unit value of the redundant at C to the primary structure and determine the displacement at C, δCC (see Fig. 9.3). This quantity is designated as ΔC, 0. Figure 9.3 illustrates this notation. 3. Next, we apply a unit value of the redundant force at C to the primary structure and compute the displacement at C. We denote this displacement as δCC (see Fig. 9.4). 4. We obtain the total displacement at C of the primary structure by superimposing the displacement profiles generated by the external loading and the reaction force at C. ΔC primary ¼ ΔC, 0 þ δCC RC 09:1Þ 5. The key step is to require the displacement at C of the primary structure to be equal to the displacement at C of the actual structure. Fig. 9.1 Actual structure Fig. 9.2 Primary structure 9.1 Introduction 563 Fig. 9.3 Displacements due to the external loading Fig. 9.4 Displacement due to unit value of ΔC actual ¼ ΔC primary ¼ ΔC, 0 þ δCC RC 09:2Þ Equation (9.2) is referred to as a "geometric compatibility" equation. When this equation is satisfied, the final displacement profiles for the actual and the primary structure will be identical. It follows that the force in the primary structure and the actual structure are also the same. 6. We solve the compatibility equation for the redundant force, RC ¼ ΔC, 0 δCC 09:3Þ Note that when RC is negative, the sense assumed in Fig. 9.4 needs to be reversed. 7. The last step involves computing the member forces in the actual structure. We superimpose the member forces according to the following algorithm: 09:4Þ Force ¼ Force external load þ RC ForceRC ¼1 Since the primary structure is statically determinate, all of the analysis required can be carried out using the methods presented in Chaps. 2, 3, 4, 5, and 6 is applicable as described above. Since the force method involves scaling and superimposing displacement profiles. The force method is particularly appealing for those who have a solid understanding of structural behavior, and when applied to simple structures, one can establish the sense of the redundant force through qualitative reasoning. 564 9 The Force Method Fig. 9.5 Actual structure Essentially, the same approach is followed for structures having more than one degree of indeterminacy. For example, consider the structure shown in Fig. 9.5. There are two excess vertical restraints. We obtain a primary structure by removing two of the vertical restraints. Note that there are multiple options for choosing the restraints to be removed. The only constraint is that the primary structure must be "stable." Figure 9.6 shows the different choices. Suppose we select the restraints at C and D as the redundants. We apply the external loading to the primary structure (Fig. 9.7) and determine the vertical displacements at C and D shown in Fig. 9.8. The next step involves computing the displacements corresponding to RC ¼ 1 and RD ¼ 1 and computing the corresponding displacements at C and D. Two separate displacement profiles are required. Combining the three displacement profiles (Fig. 9.9), the total displacements of the primary structure. ΔC primary ¼ ΔC, 0 þ δCC RC þ δCD RD 09:5Þ ΔD primary ¼ ΔD, 0 þ δDC RC þ δDD RD The coefficients of RC and RD are called flexibility coefficients. It is convenient to shift over to matrix notation at this point. We define ( ) ( ) ΔC, 0 RC Δ0 ¼ X¼ ΔD, 0 RD 09:6Þ " # δCC δCD flexibility matrix ¼ δ ¼ δDC δDD Using this notation; the geometric compatibility equation takes the form Δactual structure ¼ Δ0 þ δ X 09:7Þ Note that Δactual structure ¼ 0 when the supports are unyielding. Given the choice of primary structure, the flexibility coefficients are properties of the primary structure whereas Δ0 depends on the both the external loading and the primary structure.

structure. We solve (9.7) for X. 9.1 Introduction Fig. 9.6 Choices for primary structure. (a) Option 1. (b) Option 2. (c) Option 3 Fig. 9.7 Primary structure 565 566 9 The Force Method Fig. 9.8 Displacements due to external loading Fig. 9.9 Displacement due to unit values of the redundant. (a) RC ¼ 1. (b) RD ¼ 1 X ¼ δ1 Δactual structure Δ0 ð9:8Þ and then determine the member forces by superimposing the individual force states as follows: F ¼ Fexternal load þ FRC ¼1 RC þ FRD ¼1 RD ð9:9Þ The extension of this approach to an nth degree statically indeterminate structure just involves more computation since the individual matrices are now of order n. Since there are more redundant force quantities, we need to introduce a more systematic notation for the force and displacement quantities. Consider the frame structure shown in Fig. 9.10a. It is indeterminate to the third degree. One choice of primary structure is shown in Fig. 9.10b. We remove the support at D, take the reactions as the force redundants, and denote the jth redundant force as Xj and the corresponding displacement measure as Δj. 9.1 Introduction 567 Fig. 9.10 (a) Actual structure. (b) Primary structure—redundant reactions The resulting displacements of the primary structure due to the external loading and the three force redundants are expressed as Δ1 primary structure ¼ Δ1, 0 þ δ11 X1 þ δ12 X2 þ δ13 X3 Δ2 primary structure ¼ Δ2, 0 þ δ21 X1 þ δ22 X2 þ δ23 X3 ð9:10Þ Δ3 primary structure ¼ Δ3, 0 þ δ31 X1 þ δ32 X2 þ δ33 X3 The matrix form of (9.10) is Δprimary structure ¼ Δ0 þ δX ð9:11Þ where δ11 δ ¼ δ21 δ31 δ12 δ22 δ32 8 9 8 9 δ13 < X1 = < X1 = δ23 5 Δ0 ¼ X2 ; X ¼ X2 : ; : ; δ33 X3 X3 The notation for the flexibility coefficients is δjk = displacement at j due to a unit value of Xk. Note that the displacement measures may be either a translation or a rotation. A major portion of the computational effort is involved with computing the flexibility coefficients using the Principle of Virtual Forces. The matrix form of the equation (9.7) is generic, i.e., it is applicable for all statically indeterminate structures; one just has to establish the appropriate form for Δ0 and δ. Other possible choices of primary structures are shown in Fig. 9.11. We retain the two fixed supports, but cut the structure at an arbitrary interior point (Fig. 9.11a). The redundants are taken as the internal forces (axial, shear, and moment) at the cut section. The flexibility coefficients are now interpreted as the relative displacements of the adjacent cross sections (e.g., spreading, sliding, relative rotation). Another choice involves removing excess reactions as in Fig. 9.11b. For multi-bay multistory frames, one needs to work with internal force redundants since removing fixed supports is not sufficient to reduce the structure to a statically determinate structure. Figure 9.12 illustrates this case. Multi-span beam-type structures are handled in a similar way when choosing a primary structure. Consider Fig. 9.13. One can either select certain excess reactions or work with bending moments at interior points. We prefer the latter choice since the computation of the corresponding flexibility coefficients is simpler due to the fact that the deflection profiles associated with the redundant moments are confined to adjacent spans. For truss-type structures, various cases arise. The truss shown in Fig. 9.14a. One choice would be to remove sufficient supports such that the resulting structure is statically determinate (Fig. 9.14b). We can also keep the original restraints, and remove some members, as indicated in Fig. 9.14c. Another example is shown in Fig. 9.15a. The truss has too many members and therefore the only option is to remove some of the diagonals. Figure 9.15b illustrates this approach. 568 Fig. 9.11 (a) Primary structure—redundant internal forces. (b) Primary structure—redundant reactions Fig. 9.12 (a) Actual structure. (b) Primary structure Fig. 9.13 Multi-span beam. (a) Actual structure. (b) Primary structure— redundant reactions. (c) Primary structure— redundant moments 9 The Force Method 9.2 Maxwell's Law of Reciprocal Displacements 569 Fig. 9.14 (a) Actual structure. (b) Primary structure—redundant reactions. (c) Primary structure—redundant internal forces Fig. 9.15 (a) Actual structure. (b) Primary structure—redundant internal forces 9.2 Maxwell's Law of Reciprocal Displacements The force method leads to a set of geometric compatibility equations involve the flexibility matrix, δ. One computes the elements of δ using one of the methods described in Part I, such as the Principle of Virtual Forces. The example structures presented in Sect. 9.1 have 2 or 3 force redundants. For large n, it has 6 elements. For larger n, this computation becomes too difficult to deal with manually. However, there is a very useful relationship between the elements of δ, called "Maxwell's Law," which reduces the computational effort by almost 50 %. In what follows, we introduce Maxwell's Law specialized for member systems. We consider first a simply supported beam on unyielding supports subjected to a single concentrated unit force (Fig. 9.16a). Figure 9.16a defines the geometry and notation. The deflected shape due to 570 9 The Force Method Fig. 9.16 Reciprocal loading conditions. (a) Actual loading (MA). (c) Virtual loading (δMA) (d) Actual loading (MB). (e) Virtual loading (δMB) the unit force applied at A is plotted in Fig. 9.16b. Suppose we want to determine the deflection at B due to this load applied at A. We define this quantity as δBA. Using the Principle of Virtual Forces specialized for beam bending; we apply a unit virtual force at B (see Fig. 9.16c) and evaluate the following integral: Z dx ð9:12Þ δBA ¼ MA δMB EI where MA is the moment due to the unit load applied at A, and δMB is the moment due to the virtual unit load applied at B. 9.3 Application of the Force Method to Beam-Type Structures 571 Now, suppose we want the deflection at A due to a unit load at B. The corresponding virtual force is applied at A. Z dx δAB ¼ MB δMA EI ð9:13Þ where MB is the moment due to the unit force applied at B, and δMA is the moment due to the virtual unit load at A. But since we are applying unit loads, it follows that MA ¼ δMA MB ¼ δMB ð9:14Þ and it follows that the expressions for δAB and δBA are identical. δAB ¼ δBA ð9:15Þ This identity is called Maxwell's Law. It is applicable for linear elastic structures [1]. Returning back to the compatibility equations, defined by (9.7), we note that the coupling terms, δij and δji, are equal. We say the coefficients are symmetric with respect to their subscripts and it follows that δ is symmetrical. Maxwell's Law leads to another result called Müller–Breslau Principle which is used to establish influence lines for indeterminate beams and frames. This topic is discussed in Chaps. 13 and 15. 9.3 Application of the Force Method to Beam-Type Structures We apply the theory presented in the previous section to a set of beam-type structures. For completeness, we also include a discussion of some approximate techniques for analyzing partially restrained single-span beams that are useful for analyzing frames. Example 9.1 Given: The beam defined in Fig. E9.1a. Assume I ¼ 120(10)6 mm4, L ¼ 6 m, w ¼ 30 kN/m, vB ¼ 40 mm, and E ¼ 200 GPa. Fig. E9.1a Determine: The reactions for the following cases: (i) w ¼ 30 kN/m, vB ¼ 0 (ii) w ¼ 0, vB ¼ 40 mm (iii) w ¼ 30 kN/m, vB ¼ 40 mm 572 9 The Force Method Solution: The beam is indeterminate to the first degree. We work with the primary structure shown in Fig. E9.1b. Primary structure Applying the external loading and the unit value of RB to the primary structure, leads to the deflected shapes (Figs. E9.1c and E9.1d). Fig. E9.1b Fig. E9.1c Displacement due to external loading Fig. E9.1d Displacement due to the unit values of RB The deflection terms are given in Table 3.1. ΔB, 0 ¼ δBB ¼ wL4 # 8EI L3 " 3EI Then þ" ΔB actual ¼ wL4 =8EI Δ w L L B 3 þ δBB RB þ 4 3 actual þ wL4 =8EI Δ wL L B 3 RB ∴ RB ¼ ΔB actual ¼ 3EI ð L ¼ 3EI =L3 9.3 Application of the Force Method to Beam-Type Structures 573 Case (i): For ΔB actual ¼ 0 RB ¼ 3 3 wL4 ¼ 8EI L3 ¼ 3EI ¼ 3 wL ¼ 3ð30Þ6Þ ¼ 67:5 kN " 8 8 Knowing the value of RB, we determine the remaining reactions using the static equilibrium equations. X X Fy ¼ 0 M@A ¼ 0 5 RA ¼ wL ¼ 3ð30Þ6Þ ¼ 112:5 kN " 8 8 MA ¼ wL2 ¼ 135 kN m 8 counterclockwise Case (ii): ΔB actual ¼ -vB RB ¼ 3EI 32000Þ10Þ6 120Þ10Þ6 ¼ v ¼ ð0:040Þ ¼ 13:33 kN # RB ¼ 13:33 kN # RB ¼ 3EI L3 6Þ3 Fy ¼ 0 RA ¼ 13:33 kN " 8 MA ¼ 6RB ¼ 80 kN m 8 counterclockwise Case (iii): For w 6¼ 0 and ΔB actual ¼ -vB RB ¼ wL4 =8EI 3 RB ¼ wL ¼ 3EI ¼ 3 vB ¼ 3 ¼ 67 5 13:33 ¼ 54:2 kN " 8 L 3EI L3 574 9 The Force Method The structure is linear, so we can superimpose the solutions for cases (i) and (ii). Example 9.2 Given: The beam and loading defined in Fig. E9.2a. Assume I ¼ 600 in.4, L ¼ 24 ft, and E ¼ 29,000 ksi. Fig. E9.2a Determine: The reactions for the following cases: (i) The distributed load shown (ii) The support settlement at A Solution: The beam is indeterminate to the first degree. We take the vertical reaction at B as the force unknown and compute the deflected shape due to w and RB ¼ 1 applied to the primary structure (Figs. E9.2b and E9.2c). Fig. E9.2b Deflected shape due to loading Fig. E9.2c Deflected shape due to unit value of RB 9.3 Application of the Force Method to Beam-Type Structures 575 Case (i): The distributed load shown ΔB actual ¼ ΔB, 0 þ δBB RB = ΔB, 0 þ δBB RB ¼ 0 ∴ RB ¼ ΔB, 0 δBB The deflection terms are determined using (3.34). ΔB, 0 ¼ δBB ¼ 4wL4 729EI 4L3 243EI Then 4wL4 =729EI ΔB, 0 RB ¼ ¼ 37:8 kip " RB ¼ ¼ 3 δBB 4L =243EI Knowing the value of RB, we determine the remaining reactions using the static equilibrium equations. Case (ii): The support settlement at A (Fig. E9.2d) Fig. E9.2d Displacement due to support settlement at A þ" ΔB actual ¼ ΔB, 0 þ δBB RB where δBB ¼ 4L3 =243EI ¼ 4ð24Þ3 12Þ3 ¼ 0:386 in: 243EI 243ð29,000Þ600Þ ¼ 2ð0, 4 ¼ 6A ¼ 6 in. 3 576 9 The Force Method Therefore ΔB, 0 ¼ 4 in. RB ¼ ΔB, 0 δBB ¼ 0:386 δBB Note the deflected shape and moment diagram. Example 9.3 Given: The two-span beam defined in Fig. E9.3a. Assume E is constant, L ¼ 9 m, and w ¼ 20 kN. Fig. E9.3a Determine: The reactions Solution: The beam is indeterminate to the second degree. We remove the supports at B and C, take the vertical reactions at B and C as the force redundants, and compute the deflected shapes due to w, X1 ¼ 1, and X2 ¼ 1 applied to the primary structure (Figs. E9.3b, E9.3c, E9.3d). Fig. E9.3b Deflected shape due to loading 9.3 Application of the Force Method to Beam-Type Structures 577 Fig. E9.3c Deflected shape due to X1 ¼ 1 Fig. E9.3d Deflected shape due to X2 ¼ 1 The displacements of the primary structure are expressed as: Δ1, 0 þ δ11 X1 þ δ12 X2 ¼ 0 Δ2, 0 þ δ21 X1 þ δ22 X2 ¼ 0 Noting symmetry and the deflection results listed in Table 3.1, it follows that: X1 ¼ X2 Δ1, 0 ¼ Δ2, 0 ¼ 11wL4 12EI δ11 ¼ δ22 ¼ 4L3 9EI δ21 ¼ δ12 ¼ 7L3 18EI Then 11wL4 =12EI 11 ¼ 1:1Þ20Þ9Þ ¼ 198 kN X1 ¼ X2 ¼ δ11 þ δ12 4L =9EI þ 7L3 =18EI Then, using the static equilibrium equations, we determine the remaining reactions as FY ¼ 0 R A ¼ R D ¼ 72 kN " 9.3.1 Beam with Yielding Supports We consider next the case where a beam is supported by other members. We take one case where a beam is supported by another beam or a cable. Examples are shown in Fig. 9.17. When the beam is loaded, reactions are developed, and the supporting members deform. Assuming linear elastic behavior, these restraints can be modeled as equivalent springs. 578 9 The Force Method Fig. 9.17 Beam supported by another beam. (a) Beam. (b) Cable. (c) Column Fig. 9.18 Beam supported by another beam behave as linear elastic restraints, and can be modeled as equivalent spring elements, as indicated in Fig. 9.17. We consider here the case where a vertical restraint is provided by another beam. Figure 9.18 illustrates this case. Point B is supported by beam CD which is parallel to beam AB. In this case, point B deflects when the load is applied to beam AB. One strategy is to work with a primary structure that includes both beams such as shown in Fig. 9.19. The force redundant is now a pair of selfequilibrating forces acting at B, and the corresponding displacement measure is the relative displacement between the upper and lower contact points, designated as B and B0. 9.3 Application of the Force Method to Beam-Type Structures 579 Fig. 9.19 Choice of force redundant and displacement profiles. (a) Primary structure—force redundant system. (c) Deflection due to external loading. (c) Deflection due to redundant force at B The total displacement corresponding to X1 ¼ 1 is the sum of three terms, δ11 ¼ δ11 AB þ δ11 CD L3 δ11 CD 3EI Beam CD functions as a restraint on the movement of beam AB. The downward movement of B0 is resisted by the bending action of beam CD. Assuming linear elastic behavior, this restraint can be modeled as a linear spring of stiffness k. One chooses the magnitude of k such that the spring deflection due to the load P is the same as the beam deflection. Then, it follows from Fig. 9.20 that 1 δ ¼ 11 CD kCD ð9:16Þ Since the two beams are rigidly connected at B, the net relative displacement must be zero. 1 L3 Δ1 ¼ Δ1, 0 þ X1 þ þ X1 ¼ 0 kCD ð9:17Þ kCD 3EI 580 9 The Force Method Fig. 9.20 Equivalent spring Solving (9.17) for X1 leads to (X1 ¼ Δ1, 0 L3 1 = 3EI þ 1 kCD) ð9:18Þ Note that the value of X1 depends on the stiffness of beam CD. Taking kCD ¼ 1 corresponds to assuming a rigid support, i.e., a roller support. When kCD ¼ 0, X1 ¼ 0. It follows that the bounds on X1 are 3EI 0 < X1 < Δ1, 0 L3 ð9:19Þ When the loading is uniform, Δ1, 0 ¼ wL4 # 8EI Another type of elastic restraint is provided by a cable. Consider the structure defined in Fig. 9.21a. We replace the cable with its equivalent spring, kc ¼ AcEc and work with the primary structure shown in Fig. 9.21b. Using the results derived above, and noting that Δ1 ¼ Δ1, 0 the solution is Lc ( ) 3 X1 ¼ Δ1, 0 L3 3EI 1 þ kc 9.3 Application of the Force Method to Beam-Type Structures 581 Fig. 9.21 (a) Actual structure. (b) Primary structure—force redundant system. (c) Deflection due to applied load. (d) Deflection due to unit force X1 leads to " # X1 ¼ 3 Δ1, 0 L3 = 3EI 1 þ 3EI kc L3 ð9:20Þ If kc is small with respect to 3ELb lb , the cable acts like a rigid support, i.e., X1 approaches the value 3 for a rigid support. When k1c is large with respect to 3ELb lb , the cable is flexible and provides essentially no resistance, i.e., X1 = 0. The ratio of cable to beam flexibilities is a key parameter for the behavior of this system. Cable-stayed schemes are beams supported with inclined cables. Figure 9.22a shows the case where there is just one cable. We follow essentially the same approach as described earlier 582 9 The Force Method Fig. 9.22 (a) Cablestayed scheme. (b) Force redundant. (c) Deflection due to applied load. (d) Deflection due to X1 ¼ 1. (e) Displacement components except that now only the cable is inclined. We take the cable force as the redundant and work with the primary structure shown in Fig. 9.22b. Note that the relative movement together of points B and B0 along the inclined direction. Up to this point, we have been working with vertical displacements. Now we need to project these movements on an inclined direction. We start with the displacement profile shown in Fig. 9.22c. The vertical deflection is vB0. Projecting the on the direction of the cable leads to 9.3 Application of the Force Method to Beam-Type Structures 583 Δ1, 0 ¼ vB0 sin θ0 Fig. 9.22d shows the case where X1 ¼ 1. The total movement now consists of the elongation of the cable and the displacement of the beam. δ11 ¼ δ11 BC þ δ11 AB The elongation of the cable is Lc δ11 BC ¼ AC EC k C The beam displacement follows from Fig. 9.22e. The vertical displacement is vB ¼ sin θL3B LB 2 3EI AB ¼ vB, 1 sin θ ¼ sin θ ¼ sin θ 3EI B 3EI B Requiring Δ1 ¼ 0 leads to P sin θ a LB a3 X1 ¼ 2 6 þ sin θ2 L3B =3EB I B þ LC =EC AC ð9:21Þ Finally, we express X1 in terms of the value of the vertical reaction corresponding to a rigid support at B. X1 ¼ 2 sin θ ð Þ 3 sin θ EB I B =L3B δLC =EC AC Þ Rigid support at B þ ð9:23Þ There are two geometric parameters, θ, and the ratio of IB/LB3 to AC/LC. Note that X1 varies with the angle θ. When cables are used to stiffen beams, such as for cable-stayed bridges, the optimum cable angle is approximately 45 . The effective stiffness provided by the cable degrades rapidly with decreasing θ. 584 9 The Force Method Example 9.4 Given: The structure defined in Fig. E9.4a. Assume I ¼ 400 in.4, L ¼ 54 ft, w ¼ 2.1 kip/ft, Lc ¼ 25 kip/in., and E ¼ 29,000 ksi. Determine: The reactions, the axial force in the cable. Solution: The structure is indeterminate to the first degree. We take the force in the cable as the force unknown. The geometric compatibility equation is Δ1, 0 þ δ11 jABC þ X1 ¼ 0 The deflection terms can be determined using (3.34). wL4 ¼ 729EI 3 4L ¼ 243EI Δ1, 0 ¼ 11 ABC The deflected shapes due to X1 ¼ 1 Fig. E9.4c Deflected shape due to unit value of X1 Fig. E9.4b Deflected shape due to external loading Solving for X1, leads to: 9.3 Application of the Force Method to Beam-Type Structures X1 ¼ 585 Δ1, 0 14:6 ¼ 34:26 kip " ¼ 0 δ11 ð0:386 þ 0:25Þ þ δ11 ABC The displacement at B is vB ¼ X1 34:26 ¼ 1:37 in: # ¼ 25 kc Next, we determine the remaining reactions using the static equilibrium equations. Example 9.5 Given: The structure defined in Fig. E9.5a. Assume I ¼ 200(10)6 mm4, L ¼ 18 m, P ¼ 45 kN, AC ¼ 1300 mm2, and E ¼ 200 GPa. Fig. E9.5a Determine: The forces in the cables, the reactions, and the vertical displacement at the intersection of the cable and the beam. 586 9 The Force Method (a) For θ ¼ 45 (b) θ ¼ 15 Solution: The structure is indeterminate to the second degree. We take the cable forces as the redundant and work with the primary structure defined in Fig. E9.5b. Fig. E9.5b Primary structure Next, we compute the deflected shapes due to external loading P, X1 ¼ 1, and X2 ¼ 1 applied to the primary structure (Figs. E9.5c, E9.5d, E9.5e). Fig. E9.5c External loading P Fig. E9.5d X1 ¼ 1 Fig. E9.5e X2 ¼ 1 The displacements of the primary structure due to the external loading and the two force redundants are expressed as 9.3 Application of the Force Method to Beam-Type Structures Δ1, 0 þ δ11 X1 þ δ12 X2 ¼ 0 Δ2, 0 þ δ21 X1 þ δ22 X2 ¼ 0 where δ11 ¼ δ11 jcable δ12 ¼ δ12 jBeam δ21 ¼ δ21 jBeam δ22 ¼ δ22 jBeam þ δ22 jcable þ δ11 jBeam Δ1, 0 ¼ δ11 jBeam Δ1, 0 ¼ vB, sin θ Δ2, 0 ¼ vC, sin θ δ11 jBeam ¼ vB, 1 sin θ δ12 jBeam ¼ vB, 2 sin θ δ22 jBeam ¼ vC, 2 sin θ Because of symmetry: 3 sin θ2 L3 δ11 Beam ¼ δ22 Beam ¼ vB, 1 sin θ ¼ 256EI 7 sin θ2 L3 δ12 Beam ¼ vB, 2 sin θ ¼ 768EI 11 sin θPL3 768EI LC L ¼ AC E 4 cos θAC E Δ1, 0 ¼ Δ2, 0 ¼ vB, 0 sin θ ¼ δ11 Cable ¼ δ22 Cable X1 ¼ X2 Lastly, the equations reduced to X1 ¼ X2 ¼ Δ1, 0 δ11 Beam þ δ11 Cable þ δ12 Beam 587 588 9 The Force Method (a) For θ ¼ 45 X1 ¼ X2 ¼ Δ1, 0 ¼ 43 kN δ11 Beam þ δ11 Cable þ δ12 Beam ∴ 2X1 sin θ ¼ 2ð43Þ sin 45 ¼ 60:8 kN The remaining reactions are determined using the static equilibrium equations. (b) For θ ¼ 15 X1 ¼ X2 ¼ Δ1, 0 ¼ 109:8 kN δ11 Beam þ δ11 Cable þ δ12 Beam ∴ 2X1 sin θ ¼ 2ð109:8Þ sin 15 ¼ 56:9 kN The remaining reactions are determined using the static equilibrium equations. 9.3.2 Fixed-Ended Beams We treat next the beam shown in Fig. 9.23a. The structure is fully restrained at both A and B and therefore is indeterminate to the second degree. We take as force redundants the end moments at each end. The corresponding displacement measures are the counterclockwise end rotations, θA and θB. We write the general form of the compatibility equations as we used X instead of θ to denote the displacement measures and instead of X for the force measures. 9.3 Application of the Force Method to Beam-Type Structures 589 Fig. 9.23 (a) Fixed-ended beam. (b) Primary structure. (c) External loading— displacement profile. (d) Displacement profile for MA ¼ 1. (e) Displacement profile for MB ¼ 1 θA ¼ θA,0 þ MA θAB þ MB θAB θB ¼ θB,0 þ MA θBA þ MB θBB ð9:24Þ where θA,0 and θB,0 depend on the nature of the applied loading, and the corresponding values of MA and MB are called fixed end moments. L 3EI L θBB ¼ 3EI L θAB ¼ θBA ¼ 6EI θAA ¼ We solve (9.24) for MA and MB 2EI f2θA, 0 θB, 0g L 2EI MB ¼ f2θB, 0 θA, 0 g L MA ¼ ð9:25Þ When the ends are fixed, θA ¼ θB ¼ 0, and the corresponding values of MA and MB are usually denoted as MAF and MBF 590 9 The Force Method 2EI f2θA, 0 θB, 0 g L 2EI MBF ¼ f2θB, 0 θA, 0 g L MAF ¼ ð9:26Þ Introducing this notation in (9.25), the expressions for the end moments reduce to 2EI f2θA þ θB g þ MAF L 2EI MB ¼ f2θB þ θA g þ MBF L MA ¼ ð9:27Þ We will utilize these equations in Chap. 10. Example 9.6 Fixed End Moments for Uniformly Distributed Loading Given: The uniform distributed loading applied to a fixed beam (Fig. E9.6a). Fig. E9.6a Determine: The fixed end moments. Solution: Using Table 3.1, the rotations due to the applied loading (Fig. E9.6b). Fig. E9.6b Primary structure Noting Table 3.1, the rotations are determined θA, 0 ¼ θB, 0 ¼ wL3 24EI θA, 0 ¼ wL3 24EI Substituting for the rotations in (9.26) leads to 2EI wL3 wL3 wL2 f2θA, 0 θB, 0 g MAF ¼ L 2EI 2 ¼ L 24 24EI 12 9.3 Application of the Force Method to Beam-Type Structures 591 Fig. E9.6c Deformation of primary structure due to applied load Substituting their values in (9.26) leads to 2EI wL2 wL2 f2θ0A, 0 þ θ0B, 0 g ¼ L 6 12 12 MAF 2EI wL3 wL2 wL2 F MB ¼ MBF þ 12 12 MB ¼ ¼ L 12 ¼ f2θB, 0 þ θA, 0 g MA ¼ MBF L 24EI 12 MFA = The shear and moment diagrams are plotted below. Fig. E9.6d Note that the peak positive moment for the simply supported case is +(wL2/8). Points of inflection are located symmetrically at 592 9 The Force Method L 1 1 pffiffiffiffi 0:21L x¼ 2 3 This solution applies for full fixity. When the member is part of a frame, the restraint is provided by the adjacent members, and the end moments will generally be less than the fully fixed value. Example 9.7 Fixed End Moment—Single Concentrated Force Given: A single concentrated force applied at an arbitrary point x ¼ a on the fixed beam shown in Fig. E9.7a. Fig. E9.7a Determine: The fixed end moments. Solution: We work with the primary structure shown in Fig. E9.7b. Fig. E9.7b Primary structure Using the results listed in Table 3.1, the rotations are given by Fig. E9.7c PaðL aÞð2L aÞ θA, 0 ¼ 6LEI θA, 0 ¼ PaðL aÞðL þ aÞ 6LEI θB, 0 Fig. E9.7c Deformation of primary structure due to external loading 9.3 Application of the Force Method to Beam-Type Structures 593 Substituting in (9.26) leads to MAF ¼ PaðL aÞ2 L2 MBF ¼ PðL aÞa2 L2 The critical location for maximum fixed end moment is x ¼ L/2 and the corresponding values are MAF ¼ MBF ¼ PL 8 The shear and moment diagrams are plotted below. Note that there is a 50 % reduction in peak moment due to end fixity. Results for various loadings and end conditions are summarized in Tables 9.1 and 9.2. 594 Table 9.1 Fixed end actions for fully fixed 9 The Force Method 9.3 Application of the Force Method to Beam-Type Structures 595 Table 9.2 Fixed end actions for partially fixed 9.3.3 Analytical Solutions for Multi-Span Beams Consider the two-span beam shown in Fig. 9.24a. We allow for different lengths and different moments of inertia for the spans. Our objective here is to determine analytically how the maximum positive and negative moments vary as the load moves across the total span. We choose the moment at B as the redundant. The geometric compatibility equation involves the relative rotation at B. ΔθB ¼ ΔθB, 0 þ δθBB MB ¼ 0 ð9:28Þ where ΔθB, 0 is the relative rotation of the two cross sections at B due to the applied loading, and δθBB is the relative rotation due to a pair of unit moments applied at B. Note that the δθBB term is independent of the loading. When the loading is on span AB (see Table 3.1), ΔθB, 0 ¼ P a a2 L21 6EI1 L1 596 9 The Force Method Fig. 9.24 (a) Actual structure—notation for a two-span beam. (b) Primary structure—redundant moment. (c) Displacement due to an external loading. (d) Rotation due to external loading Then ΔθB, 0 ¼ δL1 =I 1 Þ þ L2 δL2 =I 2 Þ ð9:28Þ Given the value of MB, we can determine the reactions using the static equilibrium equations. Noting (9.28), the peak moments are given by: PL1 a a2 Negative moment MB ¼ f 1 2 2 L1 L1 ð9:29Þ a a f a2 f a2 Positive moment MD ¼ PL1 1 2 1 2 L1 L1 L1 L1 where f ¼ 1 1 þ δI1 =I2 Þ δL2 =L1 Þ ð9:30Þ 9.3 Application of the Force Method to Beam-Type Structures 597 Fig. 9.25 Bending moment distribution for load on span AB We define the ratio of I to L as the "relative stiffness" for a span and denote this parameter by r. I r¼ ð9:30Þ L Using this notation, f takes the form f ¼ 1 1 þ δr1 =r2 Þ ð9:31Þ The typical bending moment diagram is plotted in Fig. 9.25. When the load is on span BC, one just has to use a different expression for ΔθB,0. Pb L2 2 ΔθB, 0 ¼ b b2 L22 6EI2 Then MB ¼ PL2 b 2 1 2 b L2 L2 ð9:32Þ where f ¼ 1 þ δr2 =r1 Þ Having MB, we can construct the moment diagram. Locating the maximum positive moment is facilitated by Fig. 9.26a. Defining the location of P as shown in Fig. 9.26a, the solution takes the following form. Results are shown in Figs. 9.8a and 9.8ab. 598 Fig. 9.26 (a) Actual structure—notation for load on span BC. (b) Primary structure—redundant moment. (c) Rotation due to external loading. 9 The Force Method Fig. 9.26 Bending moment distribution for load on span BC 9 The Force Method 9.3 Application of the Force Method to Beam-Type Structures 599 Determine: The variation of the bending moment at B with relative stiffness of the adjacent spans (r1/ r2 ¼ 0.1, 1, and 10). E9.8a Fig. E9.8a Solution: We determine the variation of the moment at B for a range of relative stiffness ratios covering the spectrum from one span being very flexible to one span being very rigid with respect to the other span using (9.29) and (9.31). Results for the individual spans are plotted in Figs. E9.8c and E9.8d. Fig. E9.8c Load on the left span (9.29) 600 9 The Force Method Fig. E9.8d Load on the right span (9.31) Example 9.9 Two-Span Continuous Beam—Uniform Loading Given: The two-span beam defined in Fig. E9.9a. Fig. E9.9a Determine: The bending moment at B. Solution: We take the negative moment at the interior support as the force redundant. The solution process is similar to that followed for the case of a concentrated load. We determine the relative rotations at B (Fig. E9.9b). Fig. E9.9b 9.3 Application of the Force Method to Beam-Type Structures 601 The various terms are (see Table 3.1) ΔθB, 0 ¼ δθBB ¼ w1 L31 w2 L32 24EI 1 24EI 2 L1 L2 þ 3EI 1 3EI 2 Requiring the relative rotation at B equal to zero leads to ΔθB, 0 w1 L21 1 þ δw2 =w1 Þ δL2 =L1 Þ2 δr1 = r2 Þ MB ¼ ¼ þ δr1 =r2 Þ δθBB 8 where r1 ¼ I1 , L1 r2 ¼ I2 L2 Suppose the span lengths are equal. Then wL2 MB ¼ ð1 þ δw2 =w1 Þ δr1 =r2 ÞÞ 8 ð1 þ δr1 =r2 ÞÞ The bending moment at B for all combinations of I1 and I2. The moment diagram is plotted in Fig. E9.9c. Fig. E9.9c Another interesting case is where w2 ¼ 0 and I1 ¼ I2. MB ¼ wL2 ¼ wL2 8 for all combinations of span lengths. MB ¼ wL 1 2 8 1 þ δL2 =L1 Þ 602 9 The Force Method Suppose L2 ¼ L1 and I1 ¼ I2, then 1 w1 L1 2 MB ¼ 4 8 Example 9.10 Two-Span Beam with Support Settlement Given: The two-span beam shown in Fig. E9.10a. The supports at B or A experience a vertical displacement downward due to settlement of the soil under the support. Fig. E9.10a Determine: The bending moment at B. Solution: We work with the primary structure shown in Fig. E9.10b. Fig. E9.10b Primary structure When the support at B moves downward an amount vB, the relative rotation of the section at B is ΔθB, 0 ¼ vB vB þ L1 L2 9.4 Application to Arch-Type Structures 603 Compatibility requires that the moment at B be such that MB ¼ ΔθB, 0 δθBB ð9:32Þ Then MB ¼ vB 1 1 þ L1 L2 1 δL1 =I1 Þ þ δL2 =I2 Þ 3E The minus sign indicates that the bending moment is of opposite sense to that assumed in Fig. E9.10b. When the properties are the same for both spans (I1 ¼ I2 and L1 ¼ L2), MB reduces to 3EI 1 MB ¼ 2 vB . L1 When the support at A moves downward an amount vA, the behavior is reversed. In this case, ΔθB, 0 ¼ vA =L1 and MB ¼ vA =L1 3EδI1 =L1 Þ δL2 =I2 ÞÞ When the properties are the same for both spans (I1 ¼ I2 and L1 ¼ L2), MB reduces to 3EI 1 MB ¼ 2 vA . 2L21 9.4 Application to Arch-Type Structures Chapter 6 introduced the topic of arch structures. The discussion was concerned with how the geometry of arch structures is affected by displacements. Some examples were presented to illustrate how arch structures carry 604 9 The Force Method Fig. 9.27 (a) Actual structure—geometry. (b) Primary structure—redundant reaction transverse loading by a combination of both axial and bending actions. This feature makes them more efficient than beam structures for long-span applications. In what follows, we extend the analytical formulation to statically indeterminate arches. We base our analysis procedure on the force method and use the same approach as described previously for frames. Our objective here is to develop a strategy for finding the geometry for which there is minimal bending moment in the arch due to a particular loading. We consider the two-hinged arch shown in Fig. 9.27a. This structure is indeterminate to the first degree. We take the horizontal reaction at the right support as the force redundant and use the Principle of Virtual Forces described in Sect. 6.5 to determine Δ1,0, the horizontal displacement at B due to the loading, and δ11 due to a unit value of X1. The general expressions for these displacement measures follow from (6.9) ð F0 V 0 ðxÞ M 0 ðxÞ δM ds Δ1, 0 ¼ δF þ δV þ GAs EI s AE ð9:32Þ δF2 δV2 δM2 δ11 ¼ þ þ ds AE GAs EI s We usually neglect the shear deformation term. Whether one can also neglect the axial deformation term depends on the arch geometry. For completeness, we will retain this term. The two internal force systems are summarized below. We assume the applied loading is uniform per projected length (Fig. 9.28). Substituting for the force terms leads to the following expressions for the displacement measures: δL 1 wL wL wx2 ð Δ1, 0 ¼ wx sin θ cos θ þ tan α sin θ ds δ11 2 ¼ 2 2 2 EI cos θ 0 1 0 L ð Þ ¼ cos θ þ tan α tan α sin θ ds 0 AE 9.4 Application to Arch-Type Structures 605 Fig. 9.28 (a) Loading on primary structure. (b) Force due to applied loading (F0, M0). (b) Force due to X1 ¼ 1 (δF, δM) Generating closed form solutions for (9.32) is difficult. We usually use the numerical integration scheme described in Sect. 3.6.6. The solution simplifies considerably when axial deformation is neglected with respect to bending deformation. One sets X1 ¼ 1 in (9.33). This leads to Δ1, 0 δL δL wL wx2 Δy dx ð Δ1, 0 ¼ EI cos θ 0 2 2 EI cos θ 0 2 ð9:35Þ 2 δΔyÞ dx ¼ ΔyÞ EI cos θ 0 Suppose Δy is chosen such that Δy ¼ wL wx2 x βM0 2 2 δ9:36Þ Then 1, Δ1, 0 ¼ δ11 β and it follows that X1 ¼ δ11 β ¼ 1 β1 1 δβM0 Þ þ δ9:39Þ Note that there is no bending moment in the arch for this value of X1. 606 9 The Force Method With this choice of geometry, the arch carries the exterior load by axial action only; there is no bending moment. This result is based on the assumption that axial deformation is negligible. In general, there will be a small amount of bending when h is not small with respect to L, i.e., when the arch is "shallow." One can still use this approximate approach for estimating the force in the tie. Example 9.11 Parabolic Arch with Uniform Vertical Loading Given: The two-hinged arch shown in Fig. E9.11a. Fig. E9.11a Determine: The centroidal axis for the arch is defined by x ¼ 2 y ¼ 4h L 1. The bending moment in the primary structure due to the uniform loading per unit length is M0 ¼ wL wx2 x 2 2. Equation (9.36) is satisfied if one takes Δy ¼ ¼ M0 , i.e., the horizontal thrust is equal to 2 h. ... This is a scale version of the wire. M0 ¼ wL2 1 x 2 y¼ 2 h 8h Then, noting (9.36), β¼ and X1 ¼ wL2 . 8h 8h wL2 9.4 Application to Arch-Type Structures 607 The total moment is the sum of M0 and the moment due to X1. M ¼ M0 yX1 ¼ wL2 wL2 y ¼0 8 8h We see that there is no bending moment in the arch. We should have anticipated this result since a uniformly loaded cable assumes a parabolic shape. By definition, a cable has no bending rigidity and therefore no moment. We can consider an arch as an inverted cable. It follows that a two-hinged uniformly loaded parabolic arch behaves like an inverted cable. Example 9.12 Approximate Solutions Given: The two-hinged arch and the loading defined in Fig. E9.12a. The integral expression for X1 is given by (9.34). Using this expression, we have Fig. E9.12a Variable depth arch Determine: An approximate expression for X1. Assume the cross section of the arch is deeper at the abutment than at the crown, and use the following approximation to define I. I¼ I0 cos θ where I0 is the moment of inertia of the cross-sectional at the crown, and θ is the slope. 608 9 The Force Method Solution: Substituting for I and ds ¼ dx, the expression for X1 reduces to ð L M0 y dx 0 X1 ¼ ð L y2 dx 0 Suppose the applied loading is a single concentrated force, P, is applied at mid-span. The corresponding terms for a symmetrical parabolic arc are: 4y x2 Δy ¼ x L L 0 L 5 PhL2 ΔyM0 dx) EI 0 0 48 EI 0 8 L 2 ΔyÞ2 dx ¼ 15 EI 0 15 EI 0 25 L P X1 ¼ 128 h Note that the horizontal reaction is related to the loading and the geometry defined in Fig. E9.13a). Example 9.13 Given: The two-hinged arch and the loading defined in Fig. E9.13a Fig. E9.13a Determine: Plot the shape of the arch which corresponds to negligible bending. Solution: Two-hinged arch is indeterminate to the first degree. We take the horizontal reaction as the force redundant (Fig. E9.13b). 9.4 Application to Arch-Type Structures 609 Fig. E9.13b Primary structure The applied loading is given by (Fig. E9.13c) 1:5 w0x Þ ¼ w0 2:5 w0x 50 0 < x 50 Fig. E9.13c We first compute the moment in the simply supported beam spanning AB are dV 1:5 2 ¼ w0x Þ V ¼ w0 2:5w x þ C1 dx 100 dM 2:5 2 1:5 3 M ¼ w x C1 x þ C2 dx 2 300 Enforcing the boundary conditions, M ¼ 0 at x ¼ 0 M ¼ 0 x ¼ 100 which leads to C2 ¼ 0 (C1 ¼ 1:5ð100Þ 2 2:5ð100Þ 300) Finally, the moment is given by M0 ¼ 0.8w0 75x 1:25x2 þ 0:005x3 The desired shape is y ¼ ??? 610 9 The Force Method Fig. 9.29 (a) Single tie arch. (b) Multiple tied arches The M0/H is plotted below. Note that the shape is symmetric. When the abutments are inadequate to resist the horizontal thrust, different options are employed to resist the thrust. One choice is to insert a tension tie connecting the two supports, as shown in Fig. 9.29a. Another choice is to connect a set of arches in series until a solid support is reached (see Fig. 9.29b). The latter scheme is commonly used for river crossings. We take the tension in the tie as the force redundant for the tied arch. The corresponding primary structure is shown in Fig. 9.30. We just have to add the extension of the tie member to the deflection δ11. The extended form for δ11 is δ ds L δ11 ð9:38Þ ! ! δ11 ¼ y2 δ1 þ At E The expression for Δ1,0 does not change. Then, the tension in the tie is given by: ð yðM0 ds ÞÞ ð9:39Þ X1 ¼ δ11 ¼ EI 0 Δ1, 0 y2 ds ds þ At E Note that the horizontal reaction is reduced by inserting a tie member. However, now there is bending in the arch. 9.4 Application to Arch-Type Structures 611 Fig. 9.30 Choice of primary structure for a tied arch Example 9.14 Given: A parabolic arch with a tension tie (Fig. E9.14a). The arch is loaded with a uniform distributed load per horizontal projection. Consider I to be defined as I0 cos θ Fig. E9.14a Determine: The force in the tie, the horizontal reaction, and the moment in the arch. Solution: We note the structure is statically indeterminate to the first degree. We have the same loading I ¼ δ0 cos θ as in Example 9.12 which ... L Δ1, 0 ¼ wL2 ð yM0 dx EI 0 0 δ11 ¼ L δ 1 δ L 1 δ11 ¼ 8 8 wL wL3 h ¼ 2 8 8 (a) at ... Ah ¼ 15 EI 0 15 EI 0 ... cos θ 612 9 The Force Method Δ1, 0 wL2 1 ð9:40Þ X1 ¼ δ11 ¼ 8h 1 þ 15EI 0 =8At Eh2 Using this value, we determine the moment at mid-span. L M ¼ M0 hX1 2 ( ) wL2 1 δ15=8Þ I 0 =Ah2 ( ) δ15=8Þ I 0 =Ah2 L ¼ M ¼ 8 1 þ δ15=8Þ I 0 =Ah2 Note that the extension of the tension tie is introduced in bending. 9.5 Application to Frame-Type Structures Chapter 4 dealt with statically determinate frames. We focused mainly on three-hinge frames since this type of structure provides an efficient solution for enclosing a space. In this section, we analyze indeterminate frames using the force method. In the next chapter, we apply the displacement method. The analytical results generated provide the basis for comparing the structural response of determinate vs. indeterminate frames under typical loadings. 9.5.1 General Approach We consider the arbitrary-shaped single bay frame structure shown in Fig. 9.31a. We select the horizontal reaction at the right support as the force redundant. The corresponding primary structure is shown in Fig. 9.31b. The geometric compatibility equation is Δ1, 0 þ δ11 X1 ¼ 0 where Δ1 is the horizontal displacement at D. We compute δ11 and Δ1,0 with the Principle of Virtual Forces described in Sect. 4.6. The corresponding form for a plane frame specialized for negligible transverse deformation is given by (4.9) 9.5 Application to Frame-Type Structures 613 Fig. 9.31 (a) Actual structure. (b) Primary structure—redundant reaction 9.5 Application to Frame-Type Structures 613 Fig. 9.32 (a) Actual structure. (b) Primary structure—redundant reactions ð δF ¼ δX þ M F M þ δF ds δMdM EI AE s Axial deformation is small for typical non-shallow frames and we neglect it. The δ11 term is determined by applying a unit horizontal load at D and computing the bending moment at D. This term depends on the geometry and member properties, not on the external loads, and therefore has to be computed only once. The Δ1,0 term is the horizontal deflection at D due to the external loading and has to be evaluated for each loading. Different loading conditions are treated by determining Δ1,0 for each loading condition. Given these displacement terms, one determines X1 and then Δ1, 0 X1 ¼ δ11 The same approach is followed for the frame shown in Fig. 9.32. Now there are three force redundants and three geometric compatibility conditions represented by the matrix equation (9.11). A primary structure ¼ Δ0 þ δX ¼ 0 where δ is the 3 × 3 flexibility matrix. δ is independent of the loading. Most of the computational effort is involved with computing the δ0 numerically. The integration can be tedious. Sometimes numerical integration is used. However, one still has to generate the moment and displacement profiles. If the structure is symmetrical, one can reduce the computational effort by working with simplified structural models and decomposing the loading into symmetrical and anti-symmetrical components. It is very useful for estimating, in a qualitative sense, the structural response. Our primary objective is to show how these structures respond to typical loadings. We describe some approaches in Chap. 3. In what follows, we list results for different types of frames. 9.5.2 Portal Frames We consider the frame shown in Fig. 9.33a. We select the horizontal reaction at D as the force redundant. The corresponding flexibility coefficient, δ11, is determined with the Principle of Virtual Forces (see Chap. 4). 614 9 The Force Method Fig. 9.33 Portal Frame. (a) Geometry. (b) Redundant. (c) Reactions due to X1 ¼ 1. (d) Moments due to X1 ¼ 1 Reactions—gravity loading Δ1, 0 h ¼ δ11 h2 h L δ11 ¼ 3EI 1 3EI 2 ð9:40Þ This coefficient applies for all loadings. Considering the arbitrary gravity loading shown in Fig. 9.34, the expression for Δ1,0 is determined in a similar way. Δ1, 0 gravity ¼ Pab ð Þ 2EI 2 L ... 9.5 Application to Frame-Type Structures 615 Fig. 9.35 Reactions—lateral loading Lastly, we consider the lateral loading shown in Fig. 9.35. The solution for this case is Δ1,0 lateral ¼ Ph2L 2EI2 ... The shear and moment diagrams for the two loading cases are shown in Fig. 9.36. 616 9 The Force Method Fig. 9.36 Two-hinged frame (gravity loading). (a) Moment diagram. (b) Lateral loading. (d) Moment diagram 9.5.2.1 Lateral-Loading Symmetrical Portal Frame We consider first the symmetrical portal frame shown in Fig. 9.37. The structure is indeterminate to the first degree. We decompose the loading into symmetrical and anti-symmetrical components and generate the corresponding symmetrical and anti-symmetrical structural models using the material presented in Sect. 3.9. The lateral loading is anti-symmetrical. The structure is statically determinate since the bending moment at mid-span must equal zero for anti-symmetrical behavior (Fig. 9.38c). The symmetrical loading introduces no bending in the structure, only axial force in member BE. The bending moment due to the anti-symmetrical component is plotted in Fig. 9.39. Fig. 9.37 Geometry of two-hinged portal frame 9.5 Application to Frame-Type Structures 617 Fig. 9.38 Structural models. (a) Decomposition into anti-symmetrical and symmetrical loadings. (b) Anti-symmetric and symmetric models. (c) Free body diagrams of anti-symmetrical and symmetrical segments Fig. 9.39 Bending moment distribution due to anti-symmetrical lateral loading 618 9 The Force Method 9.5.2.2 Gravity-Loading Symmetrical Portal Frame We consider next the case of gravity loading applied to a two-hinged portal frame (Fig. 9.40a). Figure 9.41 shows the moment distribution. Again, we decompose the loading and treat separately the two loading cases shown in Fig. 9.40b. Fig. 9.40 (a) Two-hinged frame under gravity loading. (b) Decomposition of loading into anti-symmetrical and symmetrical components Geometry and Loading The anti-symmetrical model is statically determinate. The corresponding free body diagram and moment distribution are shown in Fig. 9.42. The symmetrical model is statically indeterminate to the first degree. We take the horizontal reaction at the right support as the force redundant and work with the primary structure shown in Fig. 9.43. The ΔD term is due to a unit value of HD. Applying the Principle of Virtual Forces specialized for only bending deformation to evaluate these terms. The corresponding expressions are δ δS ΔD, 0 ¼ ð Þ M0 δM EI δS ð9:44Þ δS ΔDD ¼ δM2 EI δS 9.5 Application to Frame-Type Structures 619 Fig. 9.41 (a) Antisymmetrical model. (b) Free body diagram—antisymmetrical segment. (c) Bending moment distribution—antisymmetrical loading Fig. 9.42 Primary structure for two-hinged frame—symmetrical loading case where M0 is the moment due to the applied loading and δM is the moment due to a unit value of HD. The moment distributions are plotted in Fig. 9.43. Evaluating the integrals leads to: P ha δ L a Þ 2 EI 2 h3 δ L Þ h h2 L ΔD, 0 ¼ ð9:46Þ δ ¼ þ 3EI 1 2EI 2 Finally, the horizontal reaction at support D is " # 1 a δL aÞ 1 HD ¼ P Lh 1 þ ð2=3Þ rg ... ð9:47Þ where rg = r-c ... 620 9 The Force Method Fig. 9.43 Bending moment distributions—symmetrical loading case. (a) Loading. (b) Moment due to X1 ¼ 1 Fig. 9.44 Final bending moment distribution Combining the results for the symmetrical and anti-symmetrical loadings results in the final bending moment distribution plotted in Fig. 9.44. The peak moments are defined by (9.48). Pa 1 M1 ¼ ... 9.5 Application to Frame-Type Structures Example 9.15 621 Two-Hinged Symmetrical Frame—Uniform Gravity Load Given: The frame shown in Fig. E9.15a Fig. E9.15a Fig. E9.15b Solution: We work with the primary structure shown in Fig. E9.15b. We only need to determine the ΔD term corresponding to the uniform loading since the δDD term is given by (9.46). The solution for HD is HD ¼ wL2 1 12h 1 þ ð2=3Þ rg =rc The bending moment is plotted in Fig. E9.15c. 622 9 The Force Method When the members AB and CD are very stiff, rg → 0, and the moment at B approaches wL2/12 which is the fixed end moment for member BC. Fig. E9.15c Bending moment distribution 9.5.2.3 Symmetrical Portal Frames with Fixed Supports We consider next the portal frame with fixed supports shown in Fig. 9.45. Because the structure is symmetrical, we decompose the loading to consist of symmetrical and antisymmetrical components. The structure is indeterminate to the third degree. (Note that it is indeterminate to the third degree but we will take advantage of the inflection point in the beam at mid-span for the anti-symmetrical case, and the vertical reaction at the hinge at that point). Figure 9.45b defines the structures corresponding to these two loading cases. Fig. 9.45 (a) Geometry. (b) Decomposition into symmetrical and antisymmetrical loading 9.5 Application to Frame-Type Structures 623 Fig. 9.46 Bending moment distribution— anti-symmetric loading Evaluating the anti-symmetrical displacement terms for the anti-symmetrical loading case, one obtains: Δ E, 0 ¼ VE ¼ ... The antisymmetrical moment distribution is plotted in Fig. 9.46. The peak values are Ph 1 ... These moments are the peak moments at the supports and the column/girder joints. Note that the peak positive moment is reduced approximately 50 % when the supports are fixed. We consider next the case where the girder is uniformly loaded. Note that the moment is now quite sensitive to the relative stiffness of column and girder. The moments at mid span, at the ends of the girder, and at the base of the columns are plotted below. Noting the moment distribution for two-hinged frame plotted in Fig. 9.47 shows the relative stiffness of column and girder properties is 0.6. Figure 9.47 shows the bending moment distribution for two-hinged frame plotted in Fig. 9.48a. 624 9 The Force Method Fig. 9.47 Moment distribution for two-hinged portal frame under gravity loading. Fig. 9.48 (a) Portal frame with fixed supports under gravity loading. (b) Moment at mid-span ... These results show that the bending moment distribution is relatively insensitive to end fixity of the base. Example ... 9.5 Application to Frame-Type Structures 625 Fig. 9.49 Bending moment distribution—gravity loading—fixed supports Fig. 9.50 Bending moment distribution—gravity loading—hinged supports 9.5.3 Pitched Roof Frames We consider next a class of portal frames where the roof is pitched, as shown in Fig. 9.51a. We choose to work with the primary structure as indicated in Fig. 9.51b. We suppose the structure is subjected to a uniform load per horizontal projection on members BC and CD. The moment distribution due to the applied loading, M0, is parabolic in x (Fig. 9.52). Taking HE ¼ 1 leads to the moment distribution shown in Fig. 9.53. It is composed of linear segments. Assuming the supports are unyielding, the flexibility coefficients are ... where L* is the length of the inclined roof members BC and CD. ... 626 9 The Force Method Fig. 9.51 (a) Pitched roof frame—definition. (b) Primary structure—redundant reaction Fig. 9.52 (a) Primary structure-external loading. (b) Bending moment distribution due to applied loading, M0 9.5 Application to Frame-Type Structures 627 Fig. 9.53 (a) Primary structure-unit load. (b) Bending moment distribution due to unit load Fig. 9.54 Distribution of moment in HE ¼ 1 ... Equation (9.57) contains the parameters h and r2, where h2 ... 628 9 The Force Method Fig. 9.55 Three-hinge solution. (a) Loading. (b) Moment diagram For convenience, we list here the relevant solution for the three-hinge frame shown in Fig. 9.55. In order to compare the solutions, we assume r2* ¼ r1, and h2 ¼ h1 in the definition equations for the peak moments. The resulting peak values are Three-hinge case (9.59): ... Two-hinge case (9.57): ... We see that the peak negative moment is reduced by approximately 20 % when the structure is reduced to a two-hinged frame. However, the positive moment is increased by a factor of 3. 9.6 Indeterminate Trusses 629 Fig. 9.56 Examples of indeterminate trusses Indeterminate truss structures are created by either adding extra members or adding extra supports, as shown in Fig. 9.56. The force redundants are taken as either reaction forces or internal forces and the corresponding displacement measures are displacements and relative displacements. We illustrate the Force Method procedure for the three-member truss shown in Fig. 9.57a. The truss is indeterminate to the first degree. The force in member BC is taken as the force redundant and 630 9 The Force Method Fig. 9.57 (a) Three-member truss. (b) Primary structure. (c) Redundant... the corresponding geometric compatibility equation is Δ1 = 0. The member forces ... Example 9.16 Given: The truss defined in Fig. E9.16a. Assume AE is constant, A ¼ 2 in.2, and E ¼ 29,000 ksi. Fig. E9.16a Determine: The member forces. 632 9 The Force Method Solution: The horizontal component of the reaction at C is taken as the force redundant (Fig. E9.16b). Fig. E9.16b Primary structure—redundant reaction We apply the geometric compatibility condition ... The corresponding forces are listed in Figs. E9.16c and E9.16d. ... The final member forces and the reactions are listed below: Member AB BC CD DA BD Bar Ray Rex Rey ... 634 9 The Force Method Example 9.17 Given: The indeterminate truss shown in Fig. E9.17a. The members have the same cross-sectional area. Assume AE is constant. A ¼ 2000 mm2, and E ¼ 200 GPa. Fig. E9.17a Determine: The member forces. Solution: The truss is indeterminate to the second degree. ... We take the forces in members AC and BD as the force redundants. ... 9.6 Indeterminate Trusses 635 ... The final member forces are determined by superimposing the results of the individual loading conditions: ... 9.7 Summary 9.7.1 Objectives • The primary objective of this chapter is to present the force method, a procedure for analyzing statically indeterminate structures that work with force quantities as the unknown variables. • Another objective is to use the force method to develop analytical solutions which are useful for identifying the key parameters that control the response and for conducting parameter sensitivity studies. 9.8 Problems 9.7.2 637 Key Factors and Concepts • The force method is restricted to linear elastic behavior. • The first step is to transform the structure to a statically determinate structure by either removing a sufficient number of redundant restraints or by introducing internal force releases at internal points. This statically determinate structure is called the primary structure. • Next one applies the external loading to the primary structure and determines the resulting displacements at the points where the restraints were removed. • For each redundant force, one applies a unit value of the force to the primary structure and determines the displacements at the constraint points. One establishes a separate equation for each constraint point. Note that all calculations are carried out on the primary structure. • Lastly, the value of the redundant force is scaled such that the total displacement is equal to the actual displacement. This requirement is expressed as Δactual ¼ Δloading þ X δunit force ÞX redundant force where the various terms are displacements at the constraint points. 9.8 Problems Problem 9.1 Determine the reactions and draw the shear and moment diagrams. Take E ¼ 29,000 ksi and I ¼ 200 in.4. Problem 9.2 ... Problem 9.3 Determine the force in spring CD. ... Problem 9.4 Given the following properties and loading, determine the reactions ... Problem 9.5 ... Problem 9.6 Use the force method to determine the forces in the cables. ... Problem 9.7 Consider the parabolic arch shown below. Assume the arch is ... Problem 9.8 Consider the semicircular arch shown below. Determine the distribution of the axial and shear forces and the bending moment. The cross-section properties are constant. Problem 9.9 ... Use a computer software system to determine the bending moment distribution and deflected shape produced by the following loadings ... Problem 9.10 ... Problem 9.11 ... Problem 9.12 ... Problem 9.13 ... Problem 9.14 Determine the peak positive and negative moments as a function of h. Consider h ¼ 10, 20, 30 ft. Problem 9.15 Using a computer software system, determine the bending moment distribution and the deflected shape ... Problem 9.16 Compare the bending moment distribution and the vertical displacement at B for the structures shown below. ... Is there any difference in behavior for the structures shown below? Answer the question without resorting to calculations. Problem 9.18 Determine the reactions and the member forces for the truss shown. ... Problem 9.19 Determine the forces in the members of the truss shown. ... Problem 9.20 ... Problem 9.21 ... Problem 9.22 ... Problem 9.23 ... Reference 1. Tauchert TR. Energy principles in structural mechanics. New York: McGraw-Hill; 1974. The Displacement Method 10 Abstract The force method, which we discussed in the previous chapter, is one of two approaches for analyzing statically indeterminate structures. In this chapter, we describe the second procedure, referred to as the displacement method. This method works with equilibrium equations expressed in terms of displacement measures that define the position of a structure, such as translations and rotations of certain points on the structure. We start by briefly introducing the method specialized for frame-type structures and then apply it to truss, beam, and frame structures. Our focus in this chapter is on developing analytical solutions and using these solutions to explain structural behavior trends. We also include a discussion of the effect of geometrically nonlinear behavior on the stiffness. Later in Chap. 12, we describe the computer-based matrix formulation of the displacement method. 10.1 Introduction The displacement method works with equilibrium equations expressed in terms of displacement measures. For truss and frame-type structures, which are composed of members connected at node points, the translations and rotations of the nodes are taken as the displacement measures. For example, the plane truss shown in Fig. 10.1a has two unknown displacements (u2, v2). The available equilibrium equations are the two force equilibrium equations for node 2. Planar beam-type structures have two displacement measures per node, the transverse displacement and the rotation. The corresponding equations are the transverse force and moment equilibrium equations for each node. Planar frame-type structures have three displacement measures per node: two translations and one rotation. One works with the force and moment equilibrium equations for each unrestrained node. In general, the number of nodal displacements will always be equal to the number of nodal equilibrium equations. For example, the plane frame shown in Fig. 10.1c has six unknown displacements (u2, v2, θ2, u3, v3, θ3). © Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_10 649 650 10 The Displacement Method Fig. 10.1 (a) Plane truss. (b) Plane beam. (c) Plane frame. (d) Plane frame has six unknown displacements (u2, v2, θ2, u3, v3, θ3). The general approach followed to generate equations can be described as follows: 1. Firstly, we decompose the structure into nodes and members. Note that the action applied by a member to the node at its end is equal in magnitude but opposite in sense to the action applied by the node on the member. 2. Secondly, we relate the end actions for a member to the displacement measures for the nodes at the ends of the member. We carry out this procedure for each member. 3. Thirdly, we establish the force equilibrium equations for each node. This step involves summing the applied external loads and the end actions for those members which are incident on the node. 4. Fourthly, we substitute for the member end actions expressed in terms of the nodal displacements. This leads to a set of equations relating the applied external loads and the nodal displacements. 5. Lastly, we solve these equations for the nodal displacements and then use these values to determine the member end actions. The total number of unknowns is now reduced by the number of displacement measures. We illustrate this procedure for some simple structures. Although this process is straightforward, the computational effort becomes considerable as the number of nodes increases. Applications of the method to various types of structure are described in the following sections. 10.2 Displacement Method Applied to a Plane Truss 651 10.2 Displacement Method Applied to a Plane Truss Consider the truss shown in Fig. 10.2. We suppose nodes 2, 3, and 4 are unyielding. We analyzed this structure with the force method in Sect. 9.6. We include it here to provide a comparison between these two methods. There are two unknown displacements, the horizontal and vertical translations of node 1. Note that the structure is statically indeterminate to the first degree, so it is a trade-off which method one selects. The first step is to develop the equations relating the member forces and the nodal displacements. We start by expressing the change in length, e, of each member in terms of the displacements for node 1. This analysis is purely geometrical and involves projecting the nodal displacement in the initial direction of the member. We define an extension as positive when the length is increased. Noting Fig. 10.3b, the geometric relations are: 10.2 Truss geometry and loading Fig. 10.3 Extension and force quantities—axial loaded member 652 10 The Displacement Method ... e = u1 cos θ1 + v1 sin θ1 ... where e0 is the initial elongation due to temperature change and fabrication error. Then, AE e ¼ AE δe0 ... We now combine the force equilibrium equations for node 1 using the free body diagram shown in Fig. 10.4. ... Solving for u1 and v1 ... This example shows that it is just as easy to apply the displacement method ... 10.3 Member Equations for Frame-Type Structures The members in frame-type structures are subjected to both bending and axial actions. The key equations for frame-type members are the equations relating the end actions to the displacements and rotations of the ends of the member. These equations play a very important role in the analysis of statically indeterminate beams and frames and also provide the basis for the matrix formulation of the displacement method for structural frames. In what follows, we develop these equations using the force method. We consider the member AB shown in Fig. 10.4a. The ends are rigidly attached to nodes. When the structure is loaded, the nodes move and the member bends. The positive sense of these quantities is defined in Fig. 10.4b, c. We refer to the shear and moment acting on the end of the member as the end actions. Our objective here is to relate the end actions (VB, MB, VA, MA) and the end displacements (vB, θB, vA, θA). We proceed as follows: Step 1. Firstly, we assume the nodes at A and B are fixed and apply the external loading to member AB. This leads to a set of end actions which we call fixed end actions. This step is illustrated in Fig. 10.5. Step 2. Next, we allow the nodes to displace. This causes additional bending of the member AB resulting in additional end actions (ΔVB, ΔMB, ΔVA, ΔMA). Figure 10.6 illustrates this notation. Step 3. Superimposing the results from steps 1 and 2 leads to the final state shown in Fig. 10.7. 654 10 The Displacement Method Fig. 10.4 Member deformation and end actions. (a) Initial geometry. (b) Deformed configuration for member AB. (c) Notation for end shear and moment. Fig. 10.5 Fixed end actions. (a) Initial. (b) Fixed end actions. 10.3 Member Equations for Frame-Type Structures 655 Fig. 10.6 Response to nodal displacements Fig. 10.7 Final state Fixed end actions for various loading cases are listed in Table 9.1. For the second step, we visualize the process as consisting of two substeps. First, we displace node B holding A fixed. Then we displace node A, holding B fixed. Combining these results result in the response shown in Fig. 10.8c. Superposition is valid since the behavior is linear. These two substeps are similar and can be analyzed using the same process. We consider first case (a) shown in Fig. 10.8a. We consider the case by considering AB to be a cantilever beam fixed at A and subjected to unknown forces, ΔVB(1) and ΔMB(1) at its free end (see Fig. 10.9a). The end displacements due to these unknown end actions must be equated to the nodal displacements vB and θB. Solving for ΔVB(1) and ΔMB(1) leads to ... 656 10 The Displacement Method Fig. 10.8 Superposition of nodal displacements. (a) Support A fixed. (b) Support B fixed. (c) Superposition Fig. 10.9 (a) Support A fixed. (b) Support B fixed. (c) Support A fixed. (b) Support B fixed. 10.3 Member Equations for Frame-Type Structures 657 Equations (10.8) and (10.9) define the end actions due to the displacements at end B. The end actions at A follow from the equilibrium conditions. ... Case (b) of Fig. 10.8 is treated in a similar manner (see Fig. 10.9b). One obtains ... Equations (10.10) and (10.11) define the end actions due to displacement of node A with B fixed. The complete solution is generated by superimposing the results for these two loading conditions and the fixed end actions.

θA þ 2 ∂vB vA Þ þ MBF L L 6EI 12EI ð0Þ θA Þ 3 ∂vB vA Þ þ MBF L L MA ¼ ΔMA þ ΔMA þ MAF ¼ L L 2EI 6EI ð0Þ 2ð0A Þ θA 3 ∂vB vA Þ þ MAF L L MBA L L 658 10 The Displacement Method and v ¼ vo 2EI n B A F 2θA þ θB 3 MAB ¼ MAB L δ10:12aÞ v ¼ vo 2EI n B A F θA þ 2θB 3 MBA ¼ þ MBA L L 2EI n B A F ¼ 2θA þ θB þ 3 vA B L L 2EI n B A F θA þ 2θB 3 ð10:12bÞ Equations (10.12a, 10.12b) are referred to as the slope-deflection equations. They are based on the sign conventions and notation defined above. 10.4 The Displacement Method Applied to Beam Structures In what follows, we first describe how the slope-deflection equations are employed to analyze horizontal beam structures, starting with two-span beams and then moving on to multi-span beams and frames. The displacement measures for beams are taken as the nodal rotations; the transverse displacements are assumed to be specified. 10.4.1 Two-Span Beams We consider the two-span beam shown in Fig. 10.10a. One starts by subdividing the beam into two beam segments and three nodes, as indicated in Figs. 10.10b, c. There are only two rotations unknowns: the rotations at nodes A and B; the rotation at node C is considered to be zero. 10.4 The Displacement Method Applied to Beam Structures 659 Fig. 10.10 Decomposition of two-span beam into beam segments and nodes. (a) Beam geometry and loading. (b) Segments and nodes. (c) Segments. (d) Reactions 660 10 The Displacement Method We apply the slope-deflection equations (10.12a) to elements AB and BC. ... Noting that the end moments are known, one can determine the end shears using the static equilibrium equations for the segments AB and BC or by using (10.12b). ... 10.4 The Displacement Method Applied to Beam Structures 661 Fig. 10.11 Beam geometry and support settlements Suppose the only external action on the above two-span beam is prescribed support settlements vA, vB, and vC as shown in Fig. 10.11. We compute the chord rotations and include these terms in the slope-deflection equations. The chord rotations are ρAB ¼ vB vA L1 ρBC ¼ vC vB L2 ... The chord rotations introduce additional end moments for each member which are added to the fixed end moments. ... Note that the solution depends on the ratio of EI to L for each span. One specifies ρ for each member, solves (10.19) for the θs, and then evaluates the end moments. 662 10 The Displacement Method Example 10.1 Given: The beam shown in Fig. E10.1a. Assume E ¼ 29,000 ksi, I ¼ 428 in.4, and L ¼ 20 ft. Fig. E10.1a Determine: The end actions and the shear and moment diagrams. Solution: First, we compute the fixed end moments by using Table 9.1. F MAB ¼ 15ð20Þ2 12 ¼ 500 kip ft ... We also need the relative member stiffnesses for each member ... 10.4 The Displacement Method Applied to Beam Structures 663 MAB ¼ 2k1 θB þ 50 MBA ¼ 2k1 2θB þ 50 MBC ¼ 2k1 2θB þ θC þ 25 MCB ¼ 2k1 2θC þ θB 25 ... θB ¼ 4.7 θC ¼ 5.357 kip ft 2k1 ... The end moments are ... Noting that the end moments are known, we can determine the end shears ... Lastly, the shear and moment diagrams are plotted below. 10.4 The Displacement Method Applied to Beam Structures 665 Example 10.2: Two-Span Symmetrical Beam—Settlement of the Supports Given: The symmetrical beam shown in Fig. E10.2a. Assume EI is constant. Take L ¼ 6 m, I ¼ 180(10)6 mm4, and E ¼ 200 kN/mm2. Fig. E10.2a Case (i): the middle support settles an amount vB ¼ 40 mm. Case (ii), the left support settles an amount vA ¼ 40 mm. Determine: The end actions, the shear and bending moment diagrams. Solution: Case (i): Support settlement at B (Fig. E10.2b) Fig. E10.2b Settlement at B Noting (10.17), the chord rotations due to settlement at B are: vA vB ¼ L L vC vB ¼ L ρAB ¼ ρBC ... The corresponding reactions are: ... One should expect that θB ¼ 0 because of symmetry. The shear and moment diagrams are plotted below. Case (ii): Support settlement at A (Fig. E10.2c) Fig. E10.2c Settlement at A Settlement at A produces chord rotation in member AB only. The chord rotation for member AB due to settlement of A is ρAB ¼ vA/L. ... Finally, the bending moment at B due to support settlement at A is: MBC ¼ MBA ¼ 60kN m counterclockwise Next, we determine the end shear forces using the static equilibrium equations for the members. ... The shear and moment diagrams are plotted below. Example 10.3: Two-Span Beam with Overhang Given: The beam shown in Fig. E10.3a. Assume EI is constant. Fig. E10.3a 669 670 10 The Displacement Method Determine: The end actions and the shear and moment diagrams. Solution: First, we compute the fixed end moments by using Table 9.1. ... Noting that θA ¼ 0 and the supports are unyielding (vA ¼ vB ¼ vC ¼ 0), the corresponding slope-deflection equations are ... The shear and moment diagrams are plotted below. 10.4.2 Multi-Span Beams In what follows, we modify the slope-deflection equations for the end members of a multi-span continuous beam when they have either a pin or roller support. Consider the three-span beam shown in Fig. 10.12a. There are three beam segments and four nodes. Since the end nodes have zero moment, we can simplify the slope-deflection equations by eliminating the end rotations. We did this in the previous examples, as part of the solution process. Now, we formalize the process and modify the slope-deflection equations before setting up the nodal moment equilibrium equations for the interior nodes. Consider member AB. The end moment at A is zero, and we use this fact to express θA in terms of θB. Starting with the expression for MAB, ... Note that the presence of a pin or roller at A reduces the rotational stiffness at B from 4EI/L to 3EI/L. Substituting for θA in the expression MBA, ... 10.4 The Displacement Method Applied to Beam Structures 673 Fig. 10.12 Three-span beam ... For member BC, we use the general unchanged form ... For member CD, we apply the same procedure used for member AB. ... MCD modified ... We return back to Fig. 10.12. If we use the modified form of the moment expressions for members AB and CD, we do not have to enforce moment equilibrium at nodes A and D since we have already employed this condition to modify the equations. Therefore we need only consider nodes B and C. Summing moments at these nodes, ... Note that the solution depends on the relative stiffness ratio, I/L, for each member. In what follows, we list the modified slope-deflection equations for an end member with a pin or roller support. ... MAB for AB (exterior pin or roller at A end): ... MBA modified ... (10.22a) (10.22b) 10.4 The Displacement Method Applied to Beam Structures Equations (10.22a, 10.22b) are referred to as the modified slope-deflection equations. Example 10.4: Three-Span Beam with Moment Releases at Both Ends Given: The three-span beam shown in Fig. E10.4a. Assume EI is constant. Fig. E10.4a Determine: The end moments. Solution: The fixed end moments are (see Table 9.1): ... We define the relative member stiffnesses for each member as ... Noting the modified slope-deflection equations (10.22a), ... Finally, the bending moments at B and C are: ... 10.4 The Displacement Method Applied to Beam Structures Using the free body diagrams below, we find the remaining end actions. 10.4 The Displacement Method Applied to Beam Structures Example 10.5: Three-Span Beam Given: The three-span beam shown in Figs. E10.5a, E10.5b, E10.5c. Fig. E10.5a Uniform load Fig. E10.5b Settlement at A Fig. E10.5c Settlement at B Determine: The end moments and draw the moment diagram for Case (i): uniform load w. No support settlement. Case (ii): no loading, Support settlement at A. Consider I and L are constants. Case (iii): no loading. Support settlement at B. Consider I and L are constants. Solution: Case (i): Uniform loading The supports are unyielding: Therefore vA ¼ vB ¼ vC ¼ vD ¼ 0. The fixed end moments due to the uniform load are (see Table 9.1) ... We use (10.22a) for members AB and CD and (10.12a) for member BC. ... The sensitivity of MBC to the ratio (L1/L2) is plotted below for various values of (I1/I2). ... Case (ii): Support settlement at A, no loading, I and L are constants ... The moment diagram is plotted below. Case (iii): Support settlement at B, no loading, I and L are constants ... 10.4 The Displacement Method Applied to Beam Structures Example 10.6: Uniformly Loaded Three-Span Symmetrical Beam—Fixed Ends Given: The three-span symmetrical fixed end beam defined in Fig. E10.6a. This model is representative of an integral bridge with very stiff abutments at the ends of the beam. Fig. E10.6a Determine: The end moments. Solution: The slope-deflection equations for unyielding supports, θA ¼ θD ¼ 0 and symmetry θB ¼ θC are ... Summing end moments at nodes B and solving for θB leads to ... Note that θB is a function of (I1/I2) and (L1/L2). The sensitivity of MBA to the ratio (L1/L2) is plotted below for various values of (I1/I2). 10.5 The Displacement Method Applied to Rigid Frames 685 The Displacement Method Applied to Rigid Frames The essential difference between the analysis of beams and frames is the choice of the nodal displacements. The variables for a beam are taken as the rotations. When there is support movement, we prescribe the nodal translation and compute the corresponding fixed end moments. In this way, the equilibrium equations always involve only rotation variables. Rigid frames are considered to be an assemblage of members rigidly connected at nodes. Since frame structures are formed by joining members at arbitrary angles, the member end translations are now unknown as well as being. When this occurs, we need to include the chord rotation terms in the slope-deflection equations, and work with both translation and rotation unknowns. Using these relations, we generate a set of equations relating the nodal translations and rotations by enforcing equilibrium for the nodes. The approach is relatively straightforward when there are not many displacement unknowns, one would usually employ a computer program which automates the generation and solution of the equilibrium equations. The term "sideway" is used to denote the case where some of the members in a structure experience chord rotation resulting in "sway" of the structure. Whether sideway occurs depends on how the members are arranged and also depends on the loading applied. For example, consider the frame shown in Fig. 10.13a. Sideway is not possible because of the horizontal restraint. The frame shown in Fig. 10.13b is symmetrical and also loaded symmetrically. Because of symmetry, there will be no sway. The frame shown in Fig. 10.13c will experience sideway. The symmetrical frame shown in Fig. 10.14 will experience sideway because of the unsymmetrical loading. All three members will experience chord rotation for the frame shown in Fig. 10.13e. When starting an analysis, one first determines whether sideway will occur in order to identify the nature of the displacement unknowns. The essential difference is that now one needs to consider force equilibrium as well as moment equilibrium of the nodes. We illustrate the analysis process with the following example. Consider the frame shown in Fig. 10.14. Under the action of the applied loading, nodes B and C will displace horizontally an amount Δ. Both members AB and CD will have chord rotation. There are 686 10 The Displacement Method Fig. 10.13 Examples of sideway three displacement unknowns θB, θC, and Δ. In general, we need to determine the end moments. Referring to Fig. 10.15. We take the positive sense of the moments to be counterclockwise. Note that this fixes the sense of the shear forces. The end moments are always positive when counterclockwise. Moment equilibrium for nodes B and C requires ... We also need to satisfy horizontal force equilibrium for the entire frame. ... The latter equation is associated with sideway. ... 10.5 The Displacement Method Applied to Rigid Frames 687 Fig. 10.14 (a) Loading. (b) Deflected shape ... Substituting for the end moments and for the end shear forces in (10.23a, 10.23b) leads to ... 689 ... (10.25) The three equations can be solved for θB, θC, and Δ. Then, the end actions are determined by substituting back into the slope-deflection equations. 10.5.1 Portal Frames: Symmetrical Loading Consider the symmetrical frame shown in Fig. 10.16. When the loading is also symmetrical, there is no sideway. ... Fig. 10.16 Portal frame—symmetrical loading. (a) Loading. (b) Deflected shape. (c) Moment diagram 690 ... The bending moment diagram is plotted in Fig. 10.16c. 10.5.2 Portal Frames: Anti-symmetrical Loading Lateral loading produces anti-symmetrical behavior, as indicated in Fig. 10.17, and chord rotation in the columns. ... The chord rotation is related to the lateral displacement by ... Fig. 10.17 Portal frame—anti-symmetric loading. (a) Loading. (b) Deflected shape. (c) Moment diagram ... A typical moment diagram is shown in Fig. 10.17c. Note the sign convention for bending moment. ... Example 10.7: Frame with No Sideway Given: The frame defined in Fig. E10.7a. Fig. E10.7a Determine: The end moments. Solution: The fixed end moments are (see Table 9.1) ... Noting that there is no sideway, ... The final bending moments ... Example 10.8: Frame with Sideway Given: The frame defined in Fig. E10.8a. Assume I and E are constant. Fig. E10.8a Determine: The end actions. Solution: The fixed end moments are (see Table 9.1) ... The chord rotations follow from the sketch below (Fig. E10.8b). ... Enforcing equilibrium at nodes B and C yields two equations, ... Summing horizontal forces for the entire frame leads to an additional equation, ... Solving these three equations, one obtains ... 10.6 The Moment Distribution Solution Procedure for Multi-span Beams 10.6.1 Introduction In the previous sections, we developed an analysis procedure for multi-span beams that involved using the slope-deflection equations to establish a set of simultaneous equations relating the nodal moments. We generated the solution by solving these equations for the rotations and then, using these values, we determined the end moments and shears. Although this approach is essentially computational, one does not gain much insight as to how the structure is responding during the solution process. This is typical of mathematical procedures which involve mainly number crunching and are ideally suited for computer-based solution schemes. The Moment Distribution Method is a solution procedure developed by Structural Engineers to solve the equilibrium equations for beam- and frame-type structures. It was originally introduced by Cross [1] and was proven to be an efficient hand-based computational scheme for beam- and frame-type structures. Its primary appeal is its conceptual simplicity. The solution is generated in an iterative manner. Each iteration cycle involves only two simple computations. Another attractive feature is that one does not have to formulate the nodal equilibrium equations expressed in terms of the nodal displacements. The method works directly with the end moments. This feature allows one to assess convergence by comparing successive values of the moments as the iterations progresses. In what follows, we illustrate the method with a series of beam-type examples. Later, we extend the method to frame-type structures. Consider the two-span beam shown in Fig. 10.18. Supports A and C are fixed, and we assume that there is no settlement at B. We assume initially that there is no rotation at B. Noting Fig. 10.19, the net unbalanced clockwise nodal moment at B is equal to the sum of the fixed end moments for the members incident on node B. ... This unbalanced moment will cause node B to rotate until equilibrium is restored. ... We use this value of θB to evaluate the incremental end moments. ... The form of the solution suggests that we define a dimensionless factor, DF, for each member as follows: ... The nodal rotation at B produces end moments in members A and C. Using the slope-deflection equations, we find ... We summarize the moment distribution procedure for this example. The steps are: 1. Determine the distribution factors at each free node (only node B in this case) 2. Determine the fixed end moments due to the applied loading and the chord rotations for the beam segments. 3. Sum the fixed end moments at node B. This sum is equal to the unbalanced moment at node B. 4. Distribute the unbalanced moment at node B. 5. Distribute one half of the incremental end moment to the other end of each member incident on node B. Executing these steps is equivalent to formulating and solving the nodal moment equilibrium equation. We illustrate this procedure with the following examples. Example 10.9: Moment Distribution Applied to a Two-Span Beam Given: The two-span beam shown in Fig. E10.9a. Fig. E10.9a Determine: The end moments. Solution: The fixed end moments are (see Table 9.1) ... It is convenient to list the end moments and distribution factors on a sketch superimposed on the multi-span beam. A typical sketch is shown below. We distribute the 22.5 kN m unbalanced moment at B and carry over the moments to A and C. After one distribution, moment equilibrium at B is restored. 10.6 The Moment Distribution Solution Procedure for Multi-span Beams 701 Since the end moments are known, one can determine the end shear forces using the static equilibrium equations for the segments. 10.6.2 Incorporation of Moment Releases at Supports We consider the case where an end member has a moment release, as shown in Fig. 10.21. We work with the modified slope-deflection equation for member AB defined in Sect. 10.3. The end moments are given by (10.22a, 10.22b) which is listed below for convenience. ... When a member has a moment release at a support then, the increment in the moment for member BA due to a rotation at B is ... We use a reduced relative rigidity factor ... Example 10.10: Two-Span Beam with a Moment Release at One End Given: The beam shown in Fig. E10.10a. Fig. E10.10a Determine: The end moments. Solution: Since member AB has a moment release, we use the modified stiffness. The modified fixed end moments are (see Table 9.2): ... The modified distribution factors for node B: ... The distribution details are listed below. Note that there is no carry-over moment to A. Using the free body diagrams below, we find the remaining end actions (Figs. E10.10b and E10.10c). 10.6.3 Moment Distribution for Multiple Free Nodes The previous examples have involved only a single free node. We now extend the method for multiple free nodes. The overall approach is the same. We just have to incorporate an iterative procedure for successively balancing the nodal moments. Consider the three-span beam shown in Fig. 10.22. Nodes B and C are free nodes. We start by computing the fixed end moments, then distribute the unbalanced nodal moments at B and C. If these moments are not equal to zero, the nodes will rotate until equilibrium is restored. Allowing a node, such as B, to rotate produces incremental end moments in members BA and BC equal to ... Similarly, a rotation at node C produces incremental end moments in segment BC and CD. ... Since there is more than one node, we start with the node having the largest unbalanced moment, distribute this moment, and carry over the distributed moment to the adjacent nodes. This operation changes the magnitudes of the remaining unbalanced moments. We select the node with the "largest" new unbalanced moment and execute a moment distribution and carry-over at this node. The solution process proceeds by successively eliminating residual nodal moments at various nodes throughout the structure. At any step, we can assess the convergence of the iteration by examining the magnitude of the remaining unbalanced moments. Usually, only a few cycles of distribution and carry-over are sufficient to obtain reasonably accurate results. Example 10.11: Moment Distribution Method Applied to a Three-Span Beam Given: The three-span beam defined in Fig. E10.11a. Fig. E10.11a Determine: The end actions. Solution: The sequence of nodal moment balancing is at the following nodes: C, B, C, B, C. We stop when the unbalanced moment is approximately less than 0.5 kip ft. The computations and distribution details are listed below. ... Using the free body diagrams below, we find the remaining end actions (Figs. E10.11b and E10.11c). Example 10.12: Example 10.11 with Moment Releases at the End Supports Given: A three-span beam with moment releases at its end supports (Fig. E10.12a). Determine: The end actions. Solution: We rework Example 10.11 with the modified fixed end moments and modified relative rigidity factors for members AB and CD. ... 10.7 Moment Distribution: Frame Structures 10.7.1 Frames: No Sideway Sideway does not occur if there is a lateral restraint. Frames with sideway are treated in a similar way as beams. The following examples illustrate the process. Example 10.13: Moment Distribution for a Frame with No Sideway Given: The frame shown in Fig. E10.13a. Fig. E10.13a Determine: The end actions. Solution: The fixed end moments are ... Using the free body diagrams below, we find the remaining end actions (Fig. E10.13c). Example 10.14: Symmetrical Two-Bay Portal Frame—Symmetrical Loading Given: The two-bay frame defined in Fig. E10.14a. Determine: The bending moment distributions and the end actions using moment distribution. Solution: We use reduced rigidity factors for the column members and no carry-over moment to the hinged ends at nodes A, D, and F (Fig. E10.14b). ... The final bending moments are listed in the sketch below. ... Note that there is no redistribution at node C because of symmetry (Fig. E10.14d). Example 10.15: Two-Bay Portal Frame—Support Settlement Given: The frame shown in Fig. E10.15a. Consider support D to experience a downward settlement of δ = 1 in. Determine: The end moments. Take E ¼ 29,000 ksi and I ¼ 2000 in.4 Solution: The support settlement produces chord rotation in members BC and CE. ... These rotations produce the following fixed end moments ... The distribution factors are listed on the following sketch (Fig. E10.15b). ... Note that the final moments are not symmetric because of the unsymmetric settlement. Example 10.16: Two-Bay Portal Frame—Temperature Increase Given: The frame shown in Fig. E10.16a. Consider members BC and CE to experience a temperature increase of ΔT. Fig. E10.16a Determine: The end moments. Solution: The top members will expand laterally, causing members AB and EF to rotate. Member CD will not rotate because of symmetry. Therefore, there will be no additional displacement (therefore no additional sideway). Noting Table 9.2, the fixed end actions corresponding to the case where there is a hinge at one end are ... The distribution details are listed below. 10.7.2 Frames with Sideway Given a frame structure, one needs to determine whether there will be chord rotation. If sideway is possible, we introduce "holding" forces applied at certain nodes to prevent this motion and carry out a conventional moment distribution based on the fixed end moments and carry-over factors. Once the fixed end moments are distributed, we can determine the member shear forces, and using these values, establish the magnitude of the holding forces. This computation is illustrated in Fig. 10.23. There is one degree of sideway, and we restrain node B. The corresponding lateral force is H1. Next, we generally neglect axial deformation for framed structures, and in this case the effect of the holding force is treated by introducing an arbitrary amount of lateral displacement. ... Fig. 10.23 (a) Frame with sideway. (b) Sideway restrained—case I ... Fig. 10.24 Sideway introduced—case II The fixed end moments due to the chord rotation produced by the horizontal displacement, Δ, are (see Table 9.1) ... Following this approach, one works only with chord rotation quantities and converts these measures into equivalent fixed end moments. The standard definition equations for the distribution factors and carry-over factors are employed to distribute the moments. Example 10.17: Portal Bent—Sideway Analysis Given: The portal frame shown in Fig. E10.17a. Determine: The end actions. Solution: Since the loading is not symmetrical, there will be sway. We need to carry out two analyses: one with the sideway restrained and the other with a specified lateral displacement. ... Case I: We introduce a lateral restraint at node B as indicated in Fig. E10.17b. ... Case II: We introduce a lateral displacement to the left equal to Δ. Figure E10.17d shows this operation. ... Example 10.18: Frame with Inclined Legs Given: The frame shown in Fig. E10.18a. Determine: The end actions. Solution: The distribution factors are listed in the sketch below (Fig. E10.18b). Case I: No sideway, and only the distributed load is applied. ... Case II: Sideway. We introduce a lateral displacement in Fig. E10.18c. ... The chord rotations of members BC and CD are determined by requiring the horizontal displacement of B and C to be equal. ... Example 10.19: Computer-Based Analysis—Frame with Inclined Legs Given: The frame shown in Fig. E10.19a. Consider a range of values for I (¼ 100, 200, and 400 in.4). Take A ¼ 20 in.2. Use computer software. Solution: The computer generated deflection profiles and the reactions and moment diagrams are listed below (Figs. E10.19b, E10.19c, E10.19d, E10.19e and E10.19f). Fig. E10.19b Deflection profile—I ¼ 100 in.4 Fig. E10.19c Deflection profile—I ¼ 200 in.4 Fig. E10.19d Deflection profile—I ¼ 400 in.4 ... 10.8 Plane Frames: Out of Plane Loading We discussed this case briefly in Chap. 4 when we dealt with statically determinate plane frame structures loaded normal to the plane such as highway signs. We extend the analysis methodology here to deal with statically indeterminate cases. Our strategy is based on the displacement method, i.e., we use generalized slope-deflection equations for the members and enforce equilibrium at the nodes. This approach is more convenient than the force method and has the additional advantage that it can be readily adopted for digital computation. 10.8.1 Slope-Deflection Equations: Out of Plane Loading Consider the prismatic member shown in Fig. 10.25a. We assume that it is loaded in the X–Z plane (note that all the previous discussions have assumed that the loading is in the X–Y plane). The relevant displacement measures for this loading are the rotation θx, the rotation θy, and the transverse displacement vz. Figure 10.25b defines the positive sense for these quantities and the corresponding end actions at A and B. Following the procedure described in Sect. 10.3, one can establish the equations relating the end actions at A and B and the end displacements at A and B. Their form is ... where GJ is the torsional rigidity for the cross section and Iy is the second moment of area with respect to y-axis. ... The remaining steps are essentially the same as for the planar case. One isolates the members and nodes and enforces equilibrium at the nodes. In what follows, we illustrate the steps involved. Consider the structure shown in Fig. 10.26. We assume there are three unknown nodal displacements, θx, θy, and vz at node B. 10.8 Plane Frames: Out of Plane Loading 731 Fig. 10.26 Plane frame and free body diagrams ... Requiring equilibrium at node B leads to the following equations: ... Lastly, we use the solution to determine the end actions for members AB and BC. ... Example 10.20: Grid Structure Given: The grid structure defined in Fig. E10.20a. The members are steel and the cross-sectional properties are constant. Take E ¼ 200 GPa, G ¼ 77 GPa, I ¼ 100(10)6 mm4, J ¼ 200(10)6 mm4. ... Determine: The displacements and member forces. Solution: The computer output data for this structure is listed below. Displacement measures at node 9: ... Member end forces at node 9: ... Member ð9Þ ... Member ð12Þ ... One checks the results by noting that the sum of the end shears at node 9 must equal the applied load of 20 kip. 10.9 Nonlinear Member Equations for Frame-Type Structures 10.9.1 Geometric Nonlinearity Although we did not mention it explicitly, when dealing with equilibrium equations, we always showed the forces acting on the initial geometry of the structure. We assumed the structure deforms as shown in Fig. 10.28 ... the geometry changes due to deformation under the load. ... are exceptions, and it is of interest to explore the consequence of accounting for geometric change when establishing the equilibrium equations. This approach is referred to as geometric nonlinear analysis since the additional geometric terms result in nonlinear equations. In what follows, we illustrate this effect for different types of structures. Consider the two-member truss shown in Fig. 10.28a. We suppose the angle θ is small, say about 15°. When a vertical force is applied, the structure deforms as shown in Fig. 10.28b. The load is resisted by the member forces generated by the deformation resulting from the displacement, v. ... For example, if θ is 15° a 1° change in the angle reduces the capacity ... Another example of geometric nonlinearity is the beam shown in Fig. 10.29a. ... Fig. 10.28 Nonlinear truss example ... Fig. 10.29 Nonlinear beam example Up to this point in the text, we have neglected the geometric term and always worked with the linearized version of the bending moment equation. ... If P is compressive, and the beam is flexible, this linearized expression is not valid and one needs to use the nonlinear form, (10.59). If P is a tensile force, the free body diagram is shown in Fig. 10.30 ... Fig. 10.30 Nonlinear beam—axial tension ... In this case, the nonlinear contribution has a stabilizing effect. Because of these observations, whenever a member is subjected to a compressive axial load, one needs to consider the potential destabilizing effect of geometric nonlinearity on the axial stiffness of the member. This type of behavior is usually referred to as "buckling." From a design perspective, buckling must be avoided. This is achieved by appropriately dimensioning the cross section and providing bracing to limit transverse displacement. In what follows, we extend the planar beam bending formulation presented in Sects. 3.5 and 3.6 to account for geometric nonlinearities. This revised formulation is applied to establish the nonlinear form of the slope-deflection equations described in Sect. 10.3. Lastly, these equations are used to investigate the nonlinear behavior of some simple frame structures. 10.9.2 Geometric Equations Accounting for Geometric Nonlinearity Figure 10.31a shows the initial and deformed positions of a differential element experiencing planar deformation in the x-y plane. The displacements are u, the axial displacement of the centroidal axis; v, the transverse displacement; and β, the rotation of the cross section. Fig. 10.31 (a) Initial and deformed positions (b) Position of tangent Figure 10.31b defines the deformed position of the centroidal axis. Assuming θ is small, the extensional strain is ... The next level of approximation is based on assuming θ is small with respect to unity. ... We assume plane sections remain plane and neglect the transverse shear deformation. ... Fig. 10.32 Orientation of deformed cross section stress–strain relations. ... If T acts at an angle θ with respect to the x-axis, ... where V is the transverse shear. Using the equilibrium requirement. 10.9.3 Equilibrium Equations Accounting for Geometric Nonlinearity The last step involves enforcing the equilibrium conditions. We work with the deformed element shown in Fig. 10.33a. ... Fig. 10.33 (a) Differential element (b) Cartesian components ... The nonlinearity is present in the transverse equilibrium equation in the form of a coupling between axial and bending actions. Lastly, the boundary conditions for the ends are: ... Case 1: Free end ... Case 2: Axial load ... Case 3: Roller support ... Case 4: Rotation restraint ... 10.9.4 Solution for Compressive Axial Load We consider here the case where the axial force is compressive (e.g., see Fig. 10.29). Taking F ¼ P constant ... This equation has constant coefficients. The homogeneous solution is ... The integration constants are determined using the boundary conditions ... Example 10.21 Given: The axially loaded beam defined in Fig. E10.21a. Fig. E10.21a Determine: The transverse displacement as a function of the axial load, P. Solution: The boundary conditions for the simply supported axially loaded beam shown in Fig. E10.21a are ... Fig. E10.21b shows the variation of the ratio of v for which P approaches Pcr, the nonlinear behavior is governed by the axial force parameter, μ. ... The effect of axial load becomes pronounced when P approaches Pcr. Euler buckling load ... 10.9.5 Nonlinear Member End Actions–End Displacement Equations In order to deal with framed structures, we need to develop the nonlinear version of the slope-deflection equations. We define the nonlinear form here using the general solution represented by (10.79). Figure 10.34a defines the positive sense for the displacement measures and the end actions. The only difference between the linear version of these equations and the nonlinear version is the presence of the axial load, P. All quantities are referred to the local member frame. We apply the boundary conditions corresponding to the end displacements to determine the integration constants. ... Fig. 10.34 Notation for nonlinear case. (a) Displacements. (b) End actions ... The limiting value of P is ... Given v, one can evaluate the end actions. The bending moments are ... Noting Fig. 10.34b, the end shear forces ... The second term in the expression for VB is due to the rotation of the chord connecting the ends of the member. This term is neglected in the linear formulation. ... The ϕ functions were introduced by Livesley [2]. Figure 10.35 shows the variation with μL. For small μL, the coefficients reduce to the corresponding linear values Fig. 10.35 ϕ functions ... For large μL, the functions behave in a nonlinear manner ... One can assume linear behavior and use these results to obtain an initial estimate for the axial load. Equations (10.86) are applicable for those structures where the ... the internal axial loads also increase, resulting in a reduction in stiffness and eventually failure of the structure. The following examples illustrate this effect. Example 10.22 Given: The portal frame shown in Fig. E10.22a. Determine: The effect of axial load on the lateral stiffness. Fig. E10.22a Solution: Consider the frame shown in Fig. E10.22a. This trend is clearly evident in the expressions for VA and VB listed in (10.86). As P increases, ϕ3 decreases, and the lateral stiffness decreases. The following examples illustrate this effect. 10.9 Nonlinear Member Equations for Frame-Type Structures 747 is the interaction between the gravity loading and the lateral displacement. Due to the compressive gravity loading, we should expect a reduction in the lateral stiffness. Eventually, the structure will become unstable. Of interest 10.9 Nonlinear Member Equations ... Noting the free body diagrams shown in Fig. E10.22b, the end conditions for member AB are ... Using these values, the expression for V B follows from Equation (10.86). ... Figure E10.22c shows how k degrades with increasing P. The P-delta term dominates this process; it leads to an 80 % reduction at the critical loading, α ¼ 1. As the load approaches this load level, the lateral stiffness approaches zero, resulting in large displacement, and eventual failure due to excessive inelastic deformation. 748 10 The Displacement Method Fig. E10.22c Example 10.23 Given: The pin-ended portal frame shown in Fig. E10.23a. Fig. E10.23a Determine: (a) The effect of axial load on the lateral stiffness. (b) The additional stiffness provided by diagonal bracing (Fig. E10.23c). Solution: Part(a) The response is anti-symmetrical so one has only to analyze member AB. Noting Fig. E10.23a, the end conditions are 10.9 Nonlinear Member Equations for Frame-Type Structures vA ¼ M A ¼ 0 749 P¼W θB ¼ 0 V B ¼ pH Setting θA ¼ 0 leads to v ¼ vi EI c h ϕ4 θ A þ ϕ2 θ B ϕ3 ¼0 h Then, noting the end conditions, and solving for θA yields ϕ EI c vB θA ¼ ϕ3 h ¼ K vB h h ϕ4 Substituting for θA, the expression for VB becomes EI c ϕ3 vB PvB VB ¼ ϕ2 vB h h 3 ϕ1 h Lastly, we require V B ¼ pH. Then H¼ EI c ϕ3 P ϕ2 vB h 3 ϕ1 h h Specializing the ϕ terms, we express H as H ¼ kvB k¼ EI c 1 ϕ3 αn 2 3EI c ϕ 2 ¼ 3 ϕk 1 k 2 þ 3 ϕ1 12 h3 3 h where α¼ P P ¼ Pcr π2 EI2 c 4h k2 ¼ Ph2 3EI c Note that k2 represents the P-delta effect on stiffness. Figure E10.23b shows that k2 dominates the stiffness reduction due to the axial compression in the columns. 750 10 The Displacement Method Fig. E10.23b Part(b) Suppose the diagonal braces shown in Fig. E10.23c are added. They provide additional stiffness which offsets the loss in stiffness due to P-delta effect. Noting the expression for VB derived above, the force H is now equal to the sum of VB and the horizontal component of the bracing force. H ¼ V B þ Fbrace cos θ ¼ ðk þ kbrace ÞvB where kbrace ¼ Abrace E Abrace E δ cos θÞ2 sin θ ð cos θÞ2 ¼ Lbrace h One selects kbrace such that kbrace ¼ Abrace E sin θ ð cos θÞ2 h 10.10 Summary 751 Fig. E10.23c etc and related problems. 10.10 Summary 10.10.1 Objectives • To describe the displacement method of analysis specialized for frame-type structures. • To develop the slope-deflection equations for planar bending of a member. • To illustrate how to apply the displacement method to planar beams and rigid frame systems using the slope deflection equations. • To formulate the moment distribution procedure and illustrate its application to indeterminate beams and rigid frames. • To derive the effect of geometric nonlinearity and to formulate the geometric nonlinear form of the slope-deflection equations. 10.10.2 Key Factors and Concepts • The displacement method works with Force Equilibrium Equations expressed in terms of displacements • The slope-deflection equations relate the end shears and moments to the end translations and rotations. Their general linear form for planar bending of a prismatic member AB is 2EI 6EI vA vB F MAB ¼ MAB f2θA þ θB g þ L L L 2EI 6EI vA vB F MBA ¼ MBA f2θB þ θA g þ L L L 6EI 12EI F V AB ¼ 2 θA þ θB þ 3 ðvA vB Þ þ VAB L L 6EI 12EI F V BA ¼ 2 θA þ θB þ 3 ðvA vB Þ þ V BA L L • Moment distribution is a numerical procedure for distributing the unbalanced nodal moments into the adjacent members based on relative stiffness. If one continued the process until the residuals are zero, one would obtain the exact solution. Normally, the process is terminated when the residuals are relatively small. • The slope-deflection equations are modified to include geometric nonlinearity. 752 10 The Displacement Method Residuals are reduced to zero. • Geometric nonlinear behavior is due to the coupling between compressive axial load and transverse displacement. It results in a loss of stiffness and leads to unstable behavior. 10.11 Problems Problem 10.1 Determine the displacements and member forces for the structure shown. Consider the following values for the areas: (a) A1 ¼ A2 ¼ A (b) A1 ¼ 2A (c) Check your results with computer-based analysis. Take E ¼ 200 GPa and A ¼ 2000 mm2 Problem 10.2 For the truss shown below, determine the displacements and member forces due to the loading shown (b) Support #1 moves as follows: u ¼ 18 in.: ! and v ¼ 12 in: " Take A ¼ 0.1 in.2, A1 ¼ 0.4 in.2, and E ¼ 29,000 ksi. 10.11 Problems 753 For the following beams and frames defined in Problems 10.3–10.18, determine the member end moments using the slope-deflection equations. Problem 10.3 Assume E ¼ 29,000 ksi, I ¼ 200 in.4, L ¼ 30 ft, vC ¼ 0.6 in. #, and w ¼ 1.2 kip/ft. Problem 10.4 (a) I1 ¼ I2 and L1 ¼ L2 (b) I1 ¼ 2I2 and L1 ¼ L2 (c) I1 ¼ 2I2 and L1 ¼ 1.2 L2 754 10 The Displacement Method Assume E ¼ 200 GPa, I2 ¼ 80(10)6 mm4, and L2 ¼ 6 m. Problem 10.5 E ¼ 29,000 ksi, I ¼ 300 in.4 Problem 10.6 Assume E ¼ 200 GPa, I ¼ 80(10)6 mm4. Problem 10.9 Assume E ¼ 29,000 ksi and I ¼ 400 in.4 Problem 10.10 Assume E ¼ 200 GPa and I ¼ 100(10)6 mm4. Problem 10.7 Assume E ¼ 29,000 ksi, I ¼ 200 in.4, L ¼ 18 ft, and w ¼ 1.2 kip/ft. 10.11 Problems Problem 10.8 Assume E ¼ 200 GPa and I ¼ 80(10)6 mm4. Problem 10.11 Problem 10.12 (a) I¼ ¼ Ib (b) Ib ¼ 2I Assume E ¼ 200 GPa, Ic ¼ 120(10)6 mm4, L ¼ 9 m, h ¼ 4 m, and w ¼ 10 kN/m. Problem 10.13 Assume E ¼ 29,000 ksi and I ¼ 200 in.4 Problem 10.14 Assume E ¼ 200 GPa and I ¼ 120(10)6 mm4 757 758 10 The Displacement Method Problem 10.18 Assume E ¼ 200 GPa and I ¼ 80(10)6 mm4. Problem 10.19 For the frame shown below, use computer software to determine the moment diagram and displacement profile. Assume E ¼ 29,000 ksi and I ¼ 300 in.4 For the following beams and frames defined in Problems 10.20–10.34, determine the member end moments using moment distribution. Problem 10.20 The loading shown (a) Support settlement of .5 in. downward at joint B in addition to the loading (b) Check your results with computer-based analysis. Take E ¼ 29,000 ksi, I ¼ 300 in.4 10.11 Problems 759 Problem 10.21 Compute the end moments and reactions. Draw the shear and moment diagrams. Check your results with computer analysis. Assume E ¼ 200 GPa and I ¼ 80(10)6 mm4. Problem 10.22 Determine the bending moments and the reactions for the following cases. Assume EI is constant. Problem 10.23 Determine the bending moment distribution for the beam shown. Assume EI is constant. 760 10 Problem 10.24 Determine the bending moment distribution. Assume I1 ¼ 1.4I2. Problem 10.25 Determine the bending moment distribution. Problem 10.26 Solve for the bending moments. δB ¼ 0.4 in. #, E ¼ 29,000 ksi, and I ¼ 240 in.4. Problem 10.27 Determine the bending moment distribution and the deflected shape. E ¼ 29,000 ksi. (a) Take I1 ¼ I2 ¼ 1000 in.4 (b) Take I1 ¼ 1.5I1. Use computer analysis. 10.11 Problems 761 Discuss the difference in behavior between case (a) and (b). Problem 10.28 Determine the axial, shear, and bending moment distributions. Take Ig ¼ 2Ic Problem 10.29 Determine the member forces and the reactions. (a) Consider only the uniform load shown (b) Consider only the support settlement of joint D (δ ¼ 0.5 in. #) (c) Consider only the temperature increase of ΔT ¼ 80 F for member BC. E ¼ 29,000 ksi, I ¼ 300 in.4, α ¼ 6.5 106/ F 762 Problem 10.30 Ig ¼ 5Ic. 10 The Displacement Method Problem 10.31 Determine the member forces and the reactions. ICD ¼ 100 in.4 IBC ¼ 400 in.4 IAB ¼ 1.5 ICD E ¼ 29,000 ksi. 10.11 Problems 763 Problem 10.33 Solve for the bending moments. Problem 10.34 For the frame shown, determine the end moments and the reactions. Problem 10.32 Determine the bending moment distribution for the following loadings. Take Problem 10.33 Solve for the bending moments. E ¼ 200 GPa and I ¼ 200(10)6 mm4. Problem 10.35 Determine analytic expression for the rotation and end moments at B. Take I ¼ 1000 in.4, A ¼ 20 in.2 for all members, and α ¼ 1.0, 2.0, 5.0. Is there an upper limit for the end moment, MBD? 764 10 The Displacement Method Problem 10.36 Compute the end moments and horizontal displacement at A for the rigid frames shown below. Check your results for parts (c) and (d) with a computer-based analysis. Take E ¼ 200 GPa and I ¼ 120(10)6 mm4. A ¼ 10,000 mm2 for all members. Problem 10.37 Compute displacement at node C for (a) No P-delta effect (b) With P-delta effect included References 765 Take I ¼ 881 in.4 , A ¼ 24 in.2 , I ¼ 2960 in.4 , A2 ¼ 35:9 in.2 , H ¼ 100 kip, P ¼ 400 kips, L ¼ 20 ft, and h ¼ 10 ft. Problem 10.38 Generate the plots of P vs. u for varying values of P. Starting at P ¼0 and increasing to T ¼ π4EI1 . Take I ¼ 400in4 , I 2 ¼ 1, and h ¼ 10 ft. 2 References 1. Cross H. Analysis of continuous frames by distributing fixed-end moments. Trans ASCE. 1932;96:1–10, Paper 1793. 2. Livesley RK. Matrix methods of structural analysis. London: Pergamon; 1964. Approximate Methods for Estimating Forces in Statically Indeterminate Structures 11 Abstract In this chapter, we describe some approximate methods for estimating the forces in indeterminate structures. We start with multi-span beams subjected to gravity loading. Then, we consider rigid frame structures under lateral loading. For this case, we distinguish between short and tall buildings. For short buildings, we first describe the portal method, an empirical procedure, for estimating the shear forces in the columns and then present an approximate stiffness approach which is more exact but less convenient to apply. For tall buildings, we model them as beams and use beam theory to estimate the forces in the columns. With all the approximate methods, our goal is to use simple hand calculation-based methods to estimate the forces which are needed for preliminary design and also for checking computer-based analysis methods. 11.1 Introduction The internal forces in a statically indeterminate structure depend on the member cross-sectional properties. We demonstrated this dependency with the examples presented in the previous two chapters. However, in order to design a structure, one needs the internal forces. Therefore, when starting the design process, it is necessary to estimate a sufficient number of force quantities so that the structure is reduced to a statically determinate structure for which the distribution of internal forces is independent of the material properties. For simple structures, such as multi-span beams and frames, the approximations are usually introduced by assuming moment releases at certain locations. The choice of release locations is based on an understanding of the behavior of the structure for the particular loading under consideration. For indeterminate trusses, we assume the magnitude of certain forces. A typical case for a truss would be when there are two diagonals in a bay. We usually assume the transverse shear is divided equally between the two diagonals. # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_11 767 768 11.2 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Multi-span Beams: Gravity Loading 11.2.1 Basic Data-Moment Diagrams Figures 11.1, 11.2, and 11.3 show moment diagrams due to a uniform loading for a range of beam geometries and support conditions. These results were generated using the methods presented in Chaps. 9 and 10. They provide the basis for assuming the location of moment releases (points of zero moment) for different combinations of span lengths and support conditions. Fig. 11.1 Moment diagrams for single-span beams Fig. 11.2 Moment diagrams for two-span beams 11.2 Multi-span Beams: Gravity Loading 769 Fig. 11.3 Moment diagrams for three-span beams. (a) Simply supported. (b) Fixed at each end. (c) Partial loading. (d) Partial loading symmetrical Fig. 11.4 Multi-span beam 11.2.2 Quantitative Reasoning Based on Relative Stiffness Consider the multi-span beam shown in Fig. 11.4. Our objective is to estimate the peak positive and negative moments in span BC. As a first step, we estimate the end moments for this span using the moment distribution factors which are related to the relative stiffness factors for the members. We consider node B. The distribution factors for members BA and BC are as follows (see Sect. 10.6): DFBA ¼ DFBC I 1 =L1 ðI1 =L1 Þ þ ðI 2 =L2 Þ I 2 =L2 ¼ ðI1 =L1 Þ þ ðI 2 =L2 Þ ð11:1Þ Note that when I is constant for all spans, the relative stiffness parameters reduce to the inverse of the span length. Given the unbalanced moment at B, we distribute it according to 770 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. 11.5 Summary of approximate models for extreme values of L1/L2 and L3/L2. (a) Hinged model. (b) Clamped end model. (c) Clamped/hinged model ΔMBA ¼ DFBA FEMB ΔMBC ¼ DFBC FEMB ð11:2Þ We consider no carry-over movement to the other ends. If L1/l1 is small in comparison to L2/I2 , then DFBA will be small in comparison to DFBC. It follows that only a small portion of the unbalanced nodal moment at node B will be distributed to member BA. The opposite case is where I1/L1 is large in comparison to I2/L2. Now DFBC is small vs. DFBA. Essentially all of the unbalanced nodal moment is distributed to member BA. The final end moment in member BC is close to its initial value (the initial fixed end moment) since there is relatively little distribution. When I is constant for all spans, the relative stiffness parameters reduce to the inverse of the span lengths. In this case, one compares the ratio of adjacent span lengths. The limiting cases for extreme values of these ratios are listed in Fig. 11.5. 11.3 Multistory Rigid Frames: Gravity Loading Gravity type loading is usually the dominant loading for multistory frames. It consists of both dead and live loading. Consider the frame shown in Fig. 11.6. We suppose the loading is a uniform gravity load, w. Our objective here is to estimate the positive and negative moments in beam AB. 11.4 Multistory Rigid Frames: Lateral Loading 771 Fig. 11.6 Multistory frame—gravity loading One can estimate moments at the ends and at the center by assuming moment releases in the beams. Assuming moment releases at 0.1L leads to w̄0:8LÞ2 ¼ 0:08wL 2 8 0:1LÞ2 þ w̄0:1LÞ0:9LÞ ¼ 0:045wL2 MA ¼ MB ¼ w 2 Mþ center ¼ M ¼ 11.4 Multistory Rigid Frames: Lateral Loading Consider the rigid frame shown in Fig. 11.7a. Under a lateral loading, the frame develops inflection points (points where the bending moment is equal to zero) in the columns and beams. Most of the approximate methods published in the literature are based on the assumption that the inflection points occur at mid-height of the columns and mid-span of the beams, as indicated in Fig. 11.7c. This assumption, coupled with an assumption concerning how the column axial and shear forces are distributed within a story, is sufficient to allow us to compute estimates for the end moments, the axial forces, and the shear forces in the columns. In what follows, we present two different approaches for estimating the forces in the columns. The first approach (11.4.1–11.4.3) estimates the column shears in a story, and is applicable mainly for low-rise rigid frames. The second approach (11.5) estimates the axial forces in the columns. Because of the nature of the assumptions, the latter procedure is appropriate only for tall, narrow rigid frames. Both procedures are derived using the idealized model of the structure shown in Fig. 11.7c, with inflection points at mid-height of the columns and mid-span of the beams. 772 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. 11.7 Multistory rigid frame. (a) Initial position. (b) Deflected Position. (c) Assumed location of inflection points 11.4.1 Portal Method The portal method is an empirical procedure for estimating the forces in low-rise rigid frames subjected to lateral loads. In addition to assuming inflection points in the columns and beams, the shear in the exterior columns is assumed to be one-half the shear in the interior columns, which is taken to be equal for all the interior columns. We use this method to generate the first estimates of the forces. Of particular interest are the end moments in the columns. Example 11.1 Application of the Portal Method Given: The rigid frame shown in Fig. E11.1a. Fig. E11.1a 11.4 Multistory Rigid Frames: Lateral Loading 773 Determine: The axial force, shear force, and bending moment in the beams and columns using the Portal method. Solution: The portal method assumes the exterior column shear, VE, is equal to one-half the interior column shear force VI, which is taken to be equal for all the interior columns. V I ¼ V I 2 Summing the column shear in a story leads to an expression for the total story shear V T ¼ 2V E þ V I ¼ 2V I Then, 1 VI ¼ VT 2 1 VE ¼ VT 2 4 We range over the stories and generate the column shear for each story. The calculations are summarized below. Story Top Bottom VT (kN) 16 48 VI (kN) 8 24 VE (kN) 4 12 Given the column shears, one can determine the column end moments using the assumption that there are inflection points at certain locations in the columns. For this structure, since the base is pinned, the inflection points for the first story are at the base. The inflection points for the second story are taken at mid-height. The free body diagrams for the various segments are shown below along with the final results. Once the column end moments are known, we can determine the end moments and shear forces in the beams and lastly, the axial forces in the columns using equilibrium equations (Figs. E11.1b, E11.1c, E11.1d, E11.1e, E11.1f, E11.1g, E11.1h). Fig. E11.1b Shear distribution for the columns (kN) 774 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. E11.1c Bending moment distribution for the columns (kN m) Fig. E11.1d Moments at the joints (kN m) Fig. E11.1e Bending moment distribution for the beams (kN m) 11.4 Multistory Rigid Frames: Lateral Loading Fig. E11.1f Shear distribution for the beams (kN) 775 776 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. E11.1g Axial and shear forces (kN) Fig. E11.1h Reactions, shear forces, and moment distribution Example 11.2 Application of the Portal Method Given: The rigid frame shown in Fig. E11.2a. Fig. E11.2a Determine: The reactions and the bending moments in the beams and columns using the Portal method. Solution: The portal method assumes the exterior column shear, VE, is equal to one-half the interior column shear force VI. The calculations are summarized in the table below. Note that, since the base is fixed, we assume inflection points at mid-height for the first story (Figs. E11.2b, E11.2c, E11.2d, E11.2e, E11.2f, E11.2g, E11.2h). Story Top Bottom VT (kip) 5 15 VI (kip) 2.5 7.5 VE (kip) 1.25 3.75 11.4 Multistory Rigid Frames: Lateral Loading Fig. E11.2b Shear distribution for the columns (kip) Fig. E11.2c Bending moment distribution for the columns (kip ft) Fig. E11.2d Moments at the joints (kip ft) 777 Fig. E11.2e Bending moment distribution for the beams (kip ft) Fig. E11.2f Shear distribution for the beams (kip) 11.4 Multistory Rigid Frames: Lateral Loading 779 Fig. E11.2h Reactions, shear forces and moment distribution 11.4.2 Shear Stiffness Method: Low-Rise Rigid Frames In this approach, we model a frame as a set of substructures, which resist the lateral shear in the stories through shearing action. First, we idealize the frame as a rigid frame with moment releases at the midpoints of the columns and beams as shown in Fig. 11.8. We consider a segment bounded by floor i + 1 and floor i. For convenience, we assume Ib and Ic are constant in a story. We allow for different values of I for the exterior and interior columns. We define VT as the sum of the lateral loads acting on floor i + 1, and all the floors above floor i + 1. This quantity represents the total transverse shear for the story. Next, we define Δu as the relative lateral displacement between floor i and floor i + 1. We assume the floor beams are rigid with respect to axial deformation so that all points on a floor experience the same lateral displacement. Lastly, we assume the floors do not move in the vertical direction, and inverse shears are indicated in Fig. 11.9. Our objective in this section is to establish an expression for the column shear forces in a story as a function of the total transverse shear for the story. 780 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. 11.8 Low-rise frame. Fig. 11.9 (a) Idealized model for a story in a low-rise frame. (b) Sub-elements of the idealized model—low-rise frame We visualize the model to consist of the sub-elements shown in Fig. 11.9b. Each sub-element experiences the same Δu. The relative stiffness factor ΔPi for sub-element i depends on the stiffness of the element. ΔPi ¼ ki Δu For a story with n sub-elements, summing the forces over the number of sub-elements leads to δ11:3Þ 11.4 Multistory Rigid Frames: Lateral Loading V T ¼ ¼ X ki Δu Δu Noting (11.3) and (11.4), the shear carried by sub-element i is given by ! ki ki X V i ¼ V T ¼ ki ki δ11:4Þ δ11:5Þ According to (11.5), the shear in a column depends on the ratio of the shear stiffness of the corresponding sub-element to the total story shear stiffness. Using the slope-deflection equations presented in Sect. 10.3, one can derive the following expressions for sub-element stiffness factors (Fig. 11.10). Fig. 11.10 Typical sub-elements. (a) Exterior. (b) Interior Exterior Element: Upper Story 12EI CE 1 12EI CE kE ¼E ¼ 3 1 þ ð h h3 CE ¼ ðIc=hÞ=ðIb=LÞ δ11:6Þ Interior Element: Upper Story 12EI CI kI ¼ fI ¼ 3 1 þ ð ð1=2ÞðI CI =hÞ=ðIb =LÞ ð h h3 δ11:7Þ where the dimensionless factor (Ic/h)/(Ib/L) accounts for the flexibility of the beam. Values of kE/kI for a range of values of (ICI/h)/(Ib/L) and ICE/ICI are tabulated in (Table 11.1). Table 11.1 Stiffness ratios: upper stories kE/kI (ICI/h)/(Ib/L) ICE/ICI 0 0.25 0.5 1.0 1.5 1.5 2.0 ICE ¼ 2 ICI 0.5 0.5 0.5 0.5 0.5 ICE ¼1 ICI 0.9 0.83 0.75 0.75 0.7 0.67 782 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Noting (11.5), we observe that the ratio of the shear in the interior column to the shear in the interior column is given by V Ecol kE ¼ V Icol The derivation listed above applies for the upper stories, and needs to be modified for the bottom story. Figure 11.11 shows the idealized model used to estimate the story stiffness for the case where the base is fixed. The sub-elements are illustrated in Fig. 11.12 and the corresponding stiffness factors are defined by (11.9) and (11.10). Fig. 11.11 Transverse shear model for bottom story—fixed support Fig. 11.12 Typical subelements for bottom story. (a) Exterior. (b) Interior Exterior Element: Base Story (Fixed Support) 9 8 1 CI eh = < 1 þ 6 kE ¼ 12EI CE h3 : 1 þ 12 CIE eh ; kBE ¼ fE 12 þ 3 CIE þ 2CE 1 þ 12EI CE ¼ ð Interior Element: Base Story (Fixed Support) 9 8 1 CI eh = < 1 þ 3 δ11:9Þ 1 þ 12EI CI 12EI CI =h ¼ h3 : 1 þ 6 C : I eh ; δ11:10Þ In this case, the inflection point is at the base. When the base is hinged, we use the expressions listed in (11.11) and (11.12). In this case, we do not assume an inflection point at mid-height of the first story (Fig. 11.13). 11.4 Multistory Rigid Frames: Lateral Loading 783 Fig. 11.13 Typical subelements for bottom story—hinged support. (a) Exterior. (b) Interior Exterior Element: Base Story (Hinged Support) 3EI CE 1 3EI CE kBE ¼ fBE ¼ 3 h3 1 þ 4 CE h h3 CE δ11:11Þ Interior Element: Base Story (Hinged Support) 3EI CI 1 3EI CI kBI ¼ 3 ¼ fBI 1 þ 2 CI h h3 CI ¼ δ1=4Þδ ICI =hÞ=δIb =LÞ δ11:12Þ The ratios of shears are in Table 11.2. V Ecol ¼ kBE V Icol kBI ¼ Table 11.2 Values of kBE/kBI for a range of (ICI/h)/(Ib/L) for both hinged and fixed supports are listed in Table 11.2. Table 11.2 Stiffness ratios: lowest story (ICI/h)/(Ib/L) 0 0.25 1.0 1.5 2.0 ICE Hinged support kE kI ¼ 1/2 ICI ICE ¼ ICI 0.5 1.0 0.5 0.944 0.5 0.9 0.5 0.857 0.5 0.833 Fixed support kE kI ¼ 1/2 ICI ICE ¼ ICI 0.5 1.0 0.5 0.95 0.5 0.948 0.5 0.91 0.5 0.833 0.5 0.75 0.5 0.786 0.5 0.75 0.5 0.833 0.5 0.816 Example 11.3 Approximate Analysis Based on Shear Stiffness Method Given: The frame shown in Fig. E11.3a. 784 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. E11.3a Consider the member properties defined as cases A, B, C, and D. I Cext =h ¼ IB =L ¼ 0:31 I B =L ¼ 0:62. I =h ¼ m I C and I B ¼ 4I C) CIBext=L ¼ IIBC ¼ L ¼ 0:625. I =h ¼ m I C and I B ¼ 4I C) CIBext=L ¼ IIBC ¼ L ¼ 1:25. I =h ¼ ext 2 I C and I B ¼ 4I C) Case B: I Cext ¼ 2I C Case D: I Cext ¼ 4I C Case A: I Cext ¼ 2I C Case B: I Cext ¼ I C Case C: I Cext ¼ 2I C Case D: I Cext ¼ I C Solution: Using (11.6), (11.7), and (11.11), the sub-element stiffnesses are: Case A fE ¼ 0.762 fI ¼ 0.762 kE ¼ 0.865 fBI ¼ 0.865 kBE ¼ 0.615 kI ¼ 0.762 kE ¼ 0.808 kI fBE ¼ 0.808 kI Top story Bottom story Case B fE ¼ 0.615 fI ¼ 0.762 kE ¼ 0.762 fBE ¼ 0.865 fBI ¼ 0.762 kBI ¼ 0.881 kBI Case C fE ¼ 0.444 fI ¼ 0.615 kE ¼ 0.722 kI fBE ¼ 0.615 fBI ¼ 0.762 kBE ¼ 0.808 kBI Case D fE ¼ 0.615 fI ¼ 0.762 kE ¼ 0.5 kI fBE ¼ 0.762 kBI ¼ 0.5 kBI ¼ 0.615 kBI Noting that V E kE ¼ V I kI we express the total shear as kE V Total ¼ 2V E þ V I ¼ V I 2 þ 1 kI Once I is specified for the interior and exterior columns, one can evaluate the ratio, kE/kI, and then VI. The computations are summarized in the table below. We also list the results predicted by the portal method. Note that the portal method approximately agrees with the stiffness method when the interior column is stiffer than the exterior column. Portal method Story Top Bottom VT (kip) VE (kip) VI (kip) VE (kip) VI (kip) 12 Case A Case B Case C VI VE VI VE VE (kip) (kip) (kip) (kip) (kip) (kip) Case D VI (kip) 2 12 6 2 3 3 4 24 4 1.235 1.53 1.11 3.65 1.64 3 6 3.83 4.34 3.7 4.59 Case D VI VE (kip) (kip) 1 2 3 6 1 Portal method VI VE (kip) (kip) 11.4 Multistory Rigid Frames: Lateral Loading 785 11.4.3 Lateral Load A rigid frame resists lateral loading through bending action of the columns. When a bracing system is combined with the frame, both of these systems participate in carrying the lateral load. From a stiffness perspective, the load is distributed according to the relative stiffness, i.e., the stiffer element carries more load. For low-rise frames, the bracing system is typically much stiffer than the rigid frame and therefore the bracing system is the controlling parameter. Figure 11.14 illustrates the structural behavior for one story. Similar arrangement is used for multistory frames. Of particular interest is the approximate distribution of lateral load between the rigid frame and the brace. Fig. 11.14 Rigid frame with Bracing The individual systems are shown in Fig. 11.15. We assume all sub-elements experience the same lateral displacement Δu, and express the lateral loads carried by each structural system as Pframe ¼ kframe Δu Pbrace ¼ kbrace Δu 11:14Þ where kframe, kbrace denote the frame and brace stiffness factors, respectively. Summing the forces, we write P ¼ kframe Δu kbrace Δu Solving for Δu, ΔP þ P = Δ kframe þ kbrace Δu (in which kbrace and kframe are given in 11:15Þ 786 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures kframe P kT kbrace P ¼ kT Pframe ¼ Pbrace δ11:16Þ Also, Pframe ¼ Pbrace kframe kbrace δ11:17Þ Considering a single story, the lateral load required to introduce an inter-story lateral displacement Δu is equal to Pframe ¼ kframe Δu where kframe is computed by combining (11.11) and (11.12). X 3E 2I CE I CI kframe ¼ 3 þ intercol 1 þ δ1=4ÞδI CI =hÞ=δIb =LÞ þ 1 þ δ1=2Þδ CE =hÞ=δIb =LÞ δ11:19Þ Once the member properties are known, one can evaluate kframe. Now we need to develop a similar expression for a brace. Figure 11.16 illustrates the brace geometry. Fig. 11.16 Diagonal bracing schemes. (a) Single. (b) Chevron. (c) X Brace Typical bracing schemes are shown in Fig. 11.16. The lateral load is carried equally by the diagonal. One can show that kbrace ¼ 2AE sin θ cos θ2 ðkchevronÞ¼ ¼ 2AE sin θ cos 2 θ1 kbrace (single) kbrace (chevron)? ¼ 1 ¼ kbracechevron ¼ kbracesingle ð11:20Þ The diagonal forces reverse when the lateral load reverses direction, as occurs for wind and earthquake loading. Example 11.4 Shear Distribution 11.4 Multistory Rigid Frames: Lateral Loading 787 Given: The one-story frame defined in Fig. E11.4a. Fig. E11.4a Determine: The lateral load carried by frame and bracing systems. Solution: Using (11.19), the frame stiffness is 3EI c 2 3 kframe ¼ 3 þ ð h 1 þ δ1=4Þ2 δ1=2Þδ c =hÞ=δIb =LÞ þ 1 þ δ1=2Þδ CI =hÞ=δIb =LÞ ¼ 3E c 8 8 EI c ð ¼ 3 ¼ 12:8 3 5 3 h h The brace stiffness follows from (11.20). Note that we use only one brace. The brace forces are shown in Fig. E11.4b. 20 30 Pbrace ¼ 40 Then Pbrace ¼ 1 ð11:79Þ   ¼ 35:6 kN 4001 ð ¼ ð650ð30Þ000 ¼ 4:36 kN The force in the member becomes 3EI c 4 kE 4 ¼ 4 ¼ 4 ¼ 12 6 0:3 4=5 0:6 ¼ 3 δ2:7 ¼ 2:2 ¼ 76 Summing the forces, 2V ¼ P ΔV ¼ 3 4 ð11:7Þ ¼ 1:36 ¼ 3 ΔV ¼ 0:05 ¼ 2 kP ¼ P kT kbrace the individual forces are related by kframe Ic Pframe ¼ 17:9 Pbrace Abrace h2 Summing the forces leads to Pframe þ Pbrace ¼ P ¼ 17:9 Ic Pbrace ¼ Pbrace 1 þ Abrace h2 Ic Lastly, the individual loads are: (11.11) and (11.12). 788 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures 3EI c 4 kE ¼ 4 3 EI ¼ 45 4 ¼ 16 ¼ 0:9V 3 Ib Summing the shears, 2V 4 PV ¼ 4:36 Then V ¼ 4 k N V 0:91 kN V E ¼ 0:56 kN Note that the brace carries the major portion of the story shear. Example 11.5 Shear Force Distribution Given: The braced rigid frame defined in Fig. E11.5a. Ab ¼ 10.8 in.2, E ¼ 29, 000 ksi, and I ¼ 150 in.4. Fig. E11.5a Determine: The distribution of shear forces between the frame and the brace systems. Solution: Pframe + Pbrace ¼ 4 kip Upper floor 12 kip Lower floor Frame: Upper story sub-element: Equations (11.6) and (11.7) are used. 11.4 Multistory Rigid Frames: Lateral Loading 789 EI c EI c 20EI c 2029; 000Þ150 h3 8EI c) kframe ¼ 4 h3 ð266Þ þ 8Þ ¼ 4ð 12Þ3 29:14 kip=in: kI ¼ 3 kE ¼ 4 Base story sub-element: Equations (11.11) and (11.12) 2EI c EI c 6:4EI c h3) kframe ¼ 3 ð202Þ þ 2:4Þ ¼ 9:32 kip=in: 2:4EI c h kI ¼ 3 kE ¼ 4 Brace: kbrace ¼ 2EAb EAb 0:70ð0:8Þð29; 000Þ ¼ 113:9 kip=in: sin θ1 θ cos θ1 Þ2 ¼ 0:707 ¼ 1 ¼ 2012Þ þ h Shear distributions Pframe ¼ kframe Pbrace kbrace 29:14 Pframe ¼ 0:256Pbrace Upper floor 113:9 9:32 Pframe ¼ 0:082Pbrace 113:9 Lower floor Therefore Pframe þ Pbrace ¼ 0:256Pbrace ¼ 4) Pbrace ¼ 3:19 kip Pframe ¼ 0:82 kip Upper floor Pframe ¼ 0:082Pbrace ¼ 12) Pbrace ¼ 11:09 kip Lower floor Example 11.6 Shear Force Distribution Given: The braced rigid frame defined in Fig. E11.6a. 790 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Fig. E11.6a Determine: The required brace area Ab, to limit the story displacement to 10 mm for each story. Assume E ¼ 200 GPa. ¼ Pbrace ¼ kbrace 2EAb sin θ cos 2 θ Upper floor 16 ¼ 3 Pbrace ¼ 16 þ 32 ¼ 48 kN Upper floor 2EAb 16ð3500Þ ¼ 39:6 mm2 Δupper) Ab 0:707ð10Þð200Þ h Lower floor 48 ¼ 0:707 EAb 48ð3500Þ ¼ 118:8 mm2 Δupper) Ab 0:707ð10Þð200Þ h The value for the lower floor controls the design. ∴Ab required ¼ 118:8 mm2 11.5 High-Rise Rigid Frames: The Cantilever Method The approximate procedure described above is applicable for low-rise rigid frames, which are referred to as "shear type" frames, i.e., the floors displace laterally but do not rotate. One determines the axial forces in the columns using the shear forces in the beam and column. As the frame height increases, the frame behaves more like a cantilever beam. As illustrated in Fig. 11.17b, the floors experience both a translation and a rotation. Just as for beams, the rotational component produces axial strain in the columns. The column shears and moments are found from equilibrium considerations, given the axial forces in the columns. We describe next an approximate procedure for estimating the distribution of column axial forces in a typical story. This approach is called the "Cantilever Method." Fig. 11.17 Tall building model—lateral deflections 792 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures One normally applies this method to estimate the axial forces in the columns at the base of the building, i.e., where the bending moment due to lateral loading is a maximum. We consider the typical tall building shown in Fig. 11.18. Given the lateral load, we can determine the bending moment and transverse shear at mid-height of each story. We denote these quantities as Mi+1 and V Tiþ1 . Fig. 11.18 Tall building model (a) under lateral loading (b) Segment of building above floor i Now, we isolate a segment of the building consisting of floors i + 1, and the columns connecting these floors. Figure 11.19 shows this segment. The floors are assumed to be rigid plates, and the columns are represented as axial springs. Floor i + 1 experience a rotation, Δβ, with respect to floor i. We position a reference axis at point O and define xi as the X coordinate for spring i. The origin of the reference axis is located such that 11.5 High-Rise Rigid Frames: The Cantilever Method 793 Fig. 11.19 Column-beam model for a story bounded by floors i and i + 1 Fig. 11.20 Deformation of column i the rotation Δβ, produces no net axial force. Mi+1 is equal to the moment M i+1. Then, when E is constant, (11.21) can be interpreted as the reference axis as equivalent to the centroidal axis for the elements rotation X k xi ¼0 δ11:21Þ Note that the axial stiffness is equal to the column stiffness, ki ¼ Ai E h δ11:22Þ where Ai is the cross-sectional area and h is the column height. Then, when E is constant, (11.21) can be interpreted as X Ai xi ¼ 0 δ11:23Þ Then one can interpret the reference axis as equivalent to the centroidal axis for the column areas in the story. We suppose the floors rotate about O and define Δβ as the relative rotation between adjacent floors. The deformation introduced in spring i follows from Fig. 11.20. ei ¼ xi Δβ Fi ¼ ki ei ¼ ki xi Δβ δ11:24Þ Summing moments about O, and equating the result to the applied moment, Mi+1 results in X 2 Miþ1 ¼ ki xi Δβ δ11:25Þ Here, Mi+1 represents the moment due to the lateral loading on and above floor i + 1. We solve for Δβ and then back substitute in the expression for Fi. The result is ! Miþ1 X ki xi F i ¼ ki xi 2 X δ11:26Þ Miþ1 k i xi2 ki xi 794 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures We see that the column force distribution is proportional to the distance from the reference axis and the relative column cross-sectional areas. One does not need to specify the actual column areas, only the ratio of areas. We next illustrate this method with an example. 11.5 High-Rise Rigid Frames: The Cantilever Method 793 Fig. 11.19 Column-beam model ... Note that we have treated the floor as a rigid plate. Stiff belt-type trusses are frequently incorporated at particular floors throughout the height so that the highrise frame behaves consistent with this hypothesis. Example 11.7 Approximate analysis based on the cantilever method Given: The symmetrical 42-story plane frame shown in Fig. E11.7a. Assume the building is supported on two caissons located at the edges of the base. Determine: The axial forces in the caissons. Fig. E11.7a Solution: The Moment at the base is given by w0 H ¼ 210w0 2 2H R ¼ 58, 800w0 ft M¼ 3 R¼ This moment is resisted by the pair of caisson forces which are equivalent to a couple. 11.5 High-Rise Rigid Frames: The Cantilever Method 795 60F ¼ 8820 F ¼ 147 kip Example 11.8 Approximate Analysis Based on the Cantilever Method Given: The symmetrical plane frame shown in Fig. E11.8a. Fig. E11.8a Determine: The axial forces in the columns in the bottom story using the cantilever method. Solution: The rotational stiffness for the bottom story is Ks ¼ X ki xi2 ¼   EX EA EA ¼ 1100 h h The bending moment at mid-height of the first story is 2550 kip ft. Then, substituting for Mi+1 and ki in (11.26) leads to the axial forces in the columns (Fig. E11.8b). 796 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures AE 2550 F1 ¼ 30Þ 2550 ¼ 23:2 kip # h 1100AE=h ¼ 255 2550 F1 ¼ 30Þ 34:8 kip # h 1100AE=h ¼ 4 2550 F2 ¼ 15Þ 23:2 kip # F4 ¼ 5 34:8 kip " Fig. E11.8b This computation is repeated for successive stories. Once all the column axial forces are known, one can compute the column shears by assuming inflection points at the midpoints of the columns and beams and applying static equilibrium conditions. The procedure is similar to that followed in Example 11.2. 11.6 Summary 11.6.1 Objectives of the Chapter Our goals in this chapter are • To describe some approximate methods for estimating the bending moment distribution in multispan beams and multi-bay frames subjected to gravity loading. • To present approximate methods for analyzing rigid frames subjected to lateral loading. 11.6.2 Key Concepts • Reasoning in a qualitative sense about the behavior using the concept of relative stiffness provides the basis for a method to estimate the bending moment distribution in multi-span beams. • Two methods are described for analyzing low-rise rigid frames. • The portal method: The portal method assumes the shear forces in the interior columns are equal to a common value, and the shears in the exterior columns are equal to ½ this value. It is an empirical-based procedure. • The shear stiffness method: The shear stiffness method uses simplified structural models to estimate the shear forces in the columns in a particular column is proportional to the relative shear stiffness of the column. It follows that a stiff column attracts more load than a flexible column. 11.7 Problems 797 • High-rise rigid frames are modeled as equivalent cantilever beams. The floor slabs are considered rigid and the bending rigidity is generated through the axial action of the columns. One uses the bending moment at the midpoint between a set of floors and determines the axial forces in the columns. According to this model, the axial force depends on the axial rigidity of the column and the distance from the centroidal axis. 11.7 Problems Problem 11.1. Estimate the bending .moment distribution for the cases listed below.. Use qualitative reasoning based on relative stiffness. Assume I as constant. Problem 11.2. Estimate the bending moment distribution. Compare the approximate and exact results. 798 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Problem 11.3. Solve Problem 11.1 cases (a) and (b) using moment distribution. Compare the approximate and exact results. Problem 11.4. Consider the multistory steel frame shown below. Determine the maximum positive and negative moments in the beam using the following approaches: 1. Assume inflection points at 0. 1 L from each end of the beams. 2. Use a computer analysis program. Assume Ig ¼ 200(10)6 mm4, Ag ¼ 16,000 mm2, Ic ¼ 100 (10)6 mm4, Ac ¼ 6000 mm2, and E ¼ 200 GPa. Problem 11.5. Estimate the axial force, shear force, and bending moment distributions. Assume Ig ¼ 2Ic 11.7 Problems 799 Problem 11.6. Members AC and FD are continuous. Estimate the bending moment distribution in AC and FD. Assume the members are pin-ended members. Compare your results with results generated using a computer software system. Assume I ¼ 100(10)6 mm4, A ¼ 6000 mm2, A pin2 ended ¼ 4000 mm2 , and E ¼ 200 GPa. Problem 11.7. Repeat Problem 11.6 assuming fixed supports at A and F. Problem 11.8. Consider the steel frame shown below. Estimate the moment at each end of each member using (a) The portal method. (b) The shear stiffness method. Take Ig ¼ 300(10)6 mm4, Ag ¼ 18,000 mm2 for all the girders, Ic ¼ 100(10)6 mm4, and Ac ¼ 6000 mm2 for all the columns. Compare your results with results generated using a computer software system. 800 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Problem 11.9. For the steel frames shown, estimate the axial force, shear force, and bending moment distributions using the portal method. Take Ic ¼ 480 in.4, Ac ¼ 15 in.2 for all the columns and Ib ¼ 600 in.4, Ab ¼ 60 in.2 for all the beams. 11.7 Problems 801 Problem 11.10. For the steel frames shown, estimate the axial force, shear force, and moments for all the members. Use the Stiffness method. Take Ic ¼ 480 in.4, Ac ¼ 40 in.2 for all the columns and Ib ¼ 600 in.4, Ab ¼ 60 in.2 for all the beams. 802 11 Approximate Methods for Estimating Forces in Statically Indeterminate Structures Problem 11.11. For the structure shown below, use the cantilever method to estimate the distribution of column areas shown. Use the same areas for the columns as specified by the cantilever method. 11.7 Problems 803 Problem 11.12. Estimate the column shears for cases (a) and (b). Compare your results with computer software results. Assume Ig ¼ 300 in.4, Ag ¼ 20 in.2, Ic ¼ 100 in.4, Ac ¼ 10 in.2, Ab ¼ 0.5 in.2, and E ¼ 29,000 ksi. Problem 11.13. Consider the rigid steel frame with bracing shown below. Estimate the column shears and brace forces. Compare your results with a computer-based solution. Take Ig ¼ 120(10)6 mm4, Ag ¼ 6000 mm2, Ic ¼ 40(10)6 mm4, Ac ¼ 4000 mm2, Ab ¼ 650 mm2, and E ¼ 200 GPa. Finite Element Displacement Method for Framed Structures 12 Abstract In this chapter, we revisit the Displacement Method for structures such as trusses, beams, and frames which are composed of member type elements. Our objective is to identify the basic steps involved in applying the Displacement Method that can be represented as computer procedures. We utilize matrix notation since it is the natural language of computation, and systematically reformulate the different steps as a sequence of matrix operations. This reformulation is referred to as the Finite Element Displacement Method. It is relatively straightforward to convert these matrix operations into computer code once one selects a computer language. 12.1 Introduction In Chap. 9 and 10, we described two methods for analyzing indeterminate structures, namely the Force and Displacement Methods. The examples that we presented were deliberately kept simple to minimize the computational effort since our objective was to demonstrate "how" the methods are applied rather than the computational details. However, one can appreciate that as a structure becomes more complex, the computational effort becomes the limiting issue for hand computation. Therefore, it is necessary to resort to computer-based procedures in order to execute the various phases of the analysis process. One needs to be familiar with commercial computer-based analysis codes since the extensive use of digital computers in structural analysis and design has revolutionized the practice of structural engineering over the past 40 years. 12.2 Key Steps of the Finite Element Displacement Method for Member Systems Chapter 10 discussed the methodology of the Displacement Method and presented examples of beam and frame structures analyzed by this method. We summarize here the key steps involved in applying # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_12 805 806 12 Finite Element Displacement Method for Framed Structures the method. The first step is to identify the nodes and represent the set of actions at a node by a set of matrix operations. Step #1. Formulate the member end force–end displacement equations Using beam theory, we express the forces acting on the ends of a member in terms of the displacement measures for the ends. These equations are referred to as member end action–end displacement equations. Their derivation is contained in Sect. 10.2. A subset of these equations are called the slope-deflection equations. In the derivation, the force and displacement components are referred to a local reference frame associated with the orientation of the member. Step #2. Select a global reference frame and transform member equations We select a common reference frame and refer both the nodal and member force and displacement measures to this common frame. This step involves shifting back and forth from member frames to the global frame and allows one to deal with structures having arbitrary geometries. Step #3. Establish the nodal force equilibrium equations We enforce force equilibrium at each node. This step involves summing the end actions for those members which are incident on the node. Then, using the member equations, we substitute for the end actions in terms of the displacement measures that correspond to the end displacements. This operation leads to a set of linear algebraic equations relating the external forces applied to the nodes and the nodal displacements. The coefficient matrix for this set is called the "System Stiffness Matrix." Step #4. Introduce displacement constraints Supports at nodes introduce constraints on certain nodal displacements. For example, if a node is fully fixed, all the displacement measures at the node are equal to zero. Introducing displacement constraints reduces the total number of displacement variables, and one works with a "reduced" set of equilibrium equations. Depending on the structure, a certain number of supports are required to prevent initial instability. Step #5. Solve the nodal force equilibrium equations for the nodal displacements. When the number of unknown displacements is large, this step is not feasible without a digital computer. Step #6. Determine member end actions We substitute the values of the nodal displacements obtained from the solution of the nodal equilibrium equations into the member force-displacement relations and solve for the member end forces. Step #7. Check on nodal force equilibrium The last step involves substituting the member end forces into the nodal equilibrium equations to check that the external nodal forces are equilibrated by the member forces. This check provides information on the reactions; it also provides a check on statics. Static discrepancy is generally related to the computational accuracy associated with solving the nodal equilibrium equations. Most computers now use double precision representation and numerical accuracy is usually not a problem. 12.3 12.3 Matrix Formulation of the Member Equations: Planar Behavior 807 Matrix Formulation of the Member Equations: Planar Behavior In what follows, we present the member equations for the two-dimensional case where bending occurs in the x–y plane. Figure 12.1 shows the end actions and displacements referred to the local frame. The x-axis coincides with the centroidal axis for the member, and the y and z axes are the principal inertia directions for the cross section. Subscripts B and A denote the positive and negative ends of the member. We use a subscript l to denote quantities that are referred to the local frame. Note that the positive sense for moment and rotation is taken as counterclockwise, i.e., from X toward Y. We derived the complete set of equations in Chap. 10. It is convenient to represent the set of end actions and end displacements as matrices defined as follows: • End Displacements UℓB 9 8 9 8 > = > < uℓB = > < uℓA ¼ vℓB ¼ vℓA UℓA ¼ vℓA ; > > ; : θA θB ð12:1Þ • End Actions PℓB 9 9 8 > = > < FℓA < FℓA > ¼ VℓB ¼ V ℓA PℓA ¼ ; > > ; : MA MB ð12:2Þ Note that the positive sense for moment and rotation is taken as counterclockwise, i.e., from X toward Y. We used a subset of these equations (10.12) to analyze bending of beams and frames. That analysis was approximate in the sense that the axial deformation of the member was neglected. Consequently, the axial forces had to be determined from the force equilibrium equations. The resulting analysis is applicable for both truss and frame type structures. The formulation is now more involved since there are now more unknowns, but this is Fig. 12.1 End forces and end displacements—local member frame—planar behavior 808 12 Finite Element Displacement Method for Framed Structures not a problem when a computer is used to solve the equations. Using the global equilibrium equations for the member, one obtains a similar set of equations for the end actions in the global frame. The complete set of planar equations for the end actions at B are: AE F ℓB ¼ uℓB uℓA Þ L 2EI 6EI 12EI F V ℓB ¼ 2 θA þ θB Þ þ 3 vℓB vℓA Þ 6EI 4EI 2EI L L MB ¼ θB þ θA þ 2 vℓB vℓA Þ L L L ð12:3Þ where FℓB , VℓB , and MB are the end actions generated by the end displacements. The fixed end actions are added to the member with the ends fixed. Using the global equilibrium equations for the member, one obtains a similar set of equations for the end actions at A. AE F ℓA ¼ uℓA uℓB Þ L 6EI 12EI V ℓA ¼ 2 θA þ θB Þ 3 vℓB vℓA Þ L L 6EI 2EI 4EI MA ¼ θB þ θA 2 vℓB vℓA Þ L L L ð12:4Þ Both sets of equations are with constant cross-sectional properties. We introduce the matrix notation defined by (12.1) and (12.2) and express the end actions in terms of the end displacements. 2 3 PℓB ¼ kℓBB UℓB þ kℓBA UℓA PℓA ¼ kℓAB UℓB þ kℓAA UℓA ð12:5Þ The coefficient matrices are 3 3 2 AE 2 AE 0 0 0 0 7 7 6 L 6 L 7 7 6 12EI 6EI 7 6 12EI 6EI 7 7 6 6 7 7 6 0 kℓBB ¼ 6 0 3 2 7 kℓBA ¼ 6 L L 7 6 L3 L2 7 6 7 7 6 6 ; F> 5 5 4 6EI 4EI 4 6EI 2EI 0 0 2 2 L L L L 2 3 2 AE 3 AE 0 0 0 0 6 L 7 7 6 L 7 6 7 6 6 12EI 6EI 7 7 6 0 12EI 6EI 7 kℓAB ¼ 6 0 7 kℓAA ¼ 6 L3 L2 7 7 6 L3 L2 7 6 ; F> 5 5 4 4 6EI 2EI 6EI 4EI 0 0 2 L L L L2 δ12:6Þ The subscripts for the member stiffness matrices refer to the end actions and displacements and ℓ denotes that they are referred to the local frame. Note that the end actions (PℓB and PℓA) depend on the external loading applied to the member. When UℓB and UℓA are known, one can determine PℓB and PℓA. The member stiffness matrices are functions of the member properties (A, I, L) and the material property E. The fixed end actions (PℓBF and PℓAF) depend on the external loading applied to the member. 12.4 Local and Global Reference Frames 809 Example 12.1: The Fixed End Actions Given: The uniformly loaded beam shown in Fig. E12.1. Fig. E12.1 Determine: The fixed end actions. Solution: The fixed end actions are determined using the Force Method described in Chap. 9 (Table 9.1). PℓB 12.4 9 8 9 8 0 0 > = > < 9 > < = wL wL ¼ 2 ¼ 2 PℓA ¼ 2 ; > > ; : > ; : wL2 wL2 12 12 Local and Global Reference Frames 20 30 Next, we consider the structure shown in Fig. 12.2. There are five nodes and four members. We assign artificially a node number. Fig. 12.2 Structure—Local and global reference frames 810 12 Finite Element Displacement Method for Framed Structures The X-axis of the local reference frame for a member is selected to coincide with the longitudinal axis, as indicated on the figure; the Y-axis is taken to be 90 counterclockwise from the X-axis. The end displacements are related to the nodal displacements. We choose the local frame defined in the previous section and define the member equations with respect to the local member frames. Figure 12.3 illustrates the situation for node two. The member end actions for a member are referred to the local frame. To establish the force equilibrium equations for the nodes, we must first select a common global reference frame and express all the force quantities with respect to this frame. When a member is arbitrarily oriented with respect to the global frame, the relations between the local member frame and the global frame must be established. Figure 12.4 shows the global reference frame and a local member frame. Starting with quantities referred to the global frame, we project them onto the local directions using trigonometric relations. One obtains ul ¼ u cos α þ v sin α vl ¼ u sin α þ v cos α θl ¼ θ ð12:7Þ Fig. 12.3 Member and global reference frames Fig. 12.3 Member and nodal forces at node 1 Fig. 12.4 Rotation of axes 12 Finite Element Displacement Method for Framed Structures 12.4 Local and Global Reference Frames 811 8 9 8 9 > < ΘmðPÞA > = ΘmPA = > Pℓ ¼ ΘmðPÞA ΘmðPÞB Pℓ ¼ > ; : Similarly the member forces are related by Pℓ ¼ ΘP ... ð12:8Þ The inverse of this transformation is Θ1 = ΘT Substituting for the local displacements and forces in (12.5), we obtain the equations in the global frame: PB ¼ kBB UB þ kBA UA PA ¼ kAB UB þ kAA UA ð12:17Þ These matrices are referred to as the member stiffness matrices in the global reference frame. 12.5 Nodal Force Equilibrium Equations Noting (12.15), the forces exerted by the members on node j are denoted by PnðjÞ ... Figure 12.6 shows the forces acting on member AB and the nodes located at each end. The actions are represented by the forces exerted by the node on the member. For convenience, we use a matrix notation. To generate the equilibrium equations at node j, one sums the external forces and the member end forces. For an arbitrary node j, X PEj ¼ PðjÞ ð12:18Þ where the summation is over all the members that are incident on node j. Note that node j corresponds to the positive end B of some members and the negative end A of other members. We utilize this observation later. We generate the complete set of nodal equilibrium equations by evaluating (12.18) for all the nodes. It is convenient to work with matrices that are partitioned according to the "natural" size of the nodal vectors. In a plane frame, the size of the nodal vectors is 3 1. For a plane truss, the size is 2 1. For a horizontal beam, the size is 2 1. We suppose there are Nn 816 12 Finite Element Displacement Method for Framed Structures nodes. Then, there will be Nn matrix equations similar in form to (12.18). We express the total set of equations as a single matrix equation, PE ¼ KU ð12:19Þ where PE is the system external nodal force vector and U is the system nodal displacement vector. 9 8 > > PE1 > > ... > ; : PENn 9 8 U1 ... U ¼ > > ; : UNn Note that the system matrices PE and U are Nn 1 and K is Nn Nn. One generates K by looping over the members and inserting the member stiffness matrices. Noting (12.18), for m ¼ 1, 2, ... , Nm: For K: kðmÞBB → K in row n , column n; kðmÞBA → K in row n , column n; kðmÞAB → K in row n, column n; kðmÞAA → K in row n, column n ð12:20Þ For PI: PðmÞFB → PI in row n; PðmÞFA → PI in row n ð12:21Þ This assembly process is called the "Direct Stiffness Method." It is generally employed by most commercial analysis software codes since it is relatively straightforward to implement. We can deduce from the assembly algorithm that the system stiffness matrix is square and symmetrical. The nonzero elements tend to be clustered in a band centered on the diagonal. 12.5 Nodal Force Equilibrium Equations Example 12.4: Assembly Process Given: The plane frame shown in Fig. E12.4a. Determine: The system stiffness matrices and the equilibrium equations. Fig. E12.4a Solution: We first number the members and nodes and then generate the member node-incidence table listed below. Member-node incidence table Member n (1) (2) (3) Negative node n+ 2 3 4 1 2 3 Positive node n+ 2 3 4 Using the member-node incidence table, we replace the member end forces and displacements with matrices with U and PI and generate the equations. Then we express the member end forces in terms of the nodal displacements. The nodal force equilibrium equations (see (12.18)) are: PE1 ¼ kð1ÞAA U1 þ kð1ÞAB U2 þ Pð1ÞFA PE2 ¼ kð1ÞBA U1 þ kð1ÞBB U2 þ kð2ÞAA U2 þ kð2ÞAB U3 þ Pð1ÞFB þ Pð2ÞFA PE3 ¼ kð2ÞBA U2 þ kð2ÞBB U3 þ kð3ÞAA U3 þ kð3ÞAB U4 þ Pð2ÞFB þ Pð3ÞFA PE4 ¼ kð3ÞBA U3 þ kð3ÞBB U4 þ Pð3ÞFB 818 12 Finite Element Displacement Method for Framed Structures Substituting for the end forces in terms of the nodal displacements, these equations expand to ... The equations are written as a single equation in terms of "system" matrices (see (12.19)). Lastly, we have listed the forms of the system matrices. Note that there are four nodes in this example, so the partitioned form of K is 4 4. The expanded size of K for this two-dimensional plane frame will be 12 12 since there are three variables per node. In this example, we chose to list all the equations first and then combine them in a single "system" equation. Normally, one would apply the algorithms defined by (12.20) and (12.21) and directly assemble the system matrices. 12.6 Introduction of Nodal Supports Introducing a support at a node corresponds to prescribing the value of certain nodal displacements. For example, a hinge prevents translation in two orthogonal directions. Fully fixity eliminates both translation and rotation. When supports are introduced, the number of displacement unknowns is decreased by the number of displacement restraints. However, each displacement constraint produces an unknown reaction so that the "total" number of unknowns (nodal displacements and nodal reactions) remains constant. In order to determine the unknown displacements, we work with a reduced set of equilibrium equations. We illustrate this process with the following example. Example 12.5: Example of Fully Fixed Supports Given: The structure shown in Fig. E12.5a. Suppose nodes one and four are fully fixed. 12.6 Introduction of Nodal Supports 819 Fig. E12.5a Determine: The reduced system matrices. Solution: The system matrices are presented in Example 12.4. We start with the complete set of nodal force equilibrium equations generated in the previous example. Nodes 1 and 4 are fully fixed. The external nodal forces PE1 and PE2 represent the reactions at these nodes. We can rearrange the order of the equations so that PE1 and PE4 are last. This step leads to two sets of equations. ... Next, we set U1 ¼ U4 ¼ 0 and partition the equations. We only need to solve the first partition, the system matrices become the following. ... We solve the first set for U2 and U3, and then evaluate the reactions using the second set. 12.6.1 Systematic Approach The systematic approach for introducing support constraints involves partitioning the system displacement vector U into two segments and also rearranging the rows and columns of P and K consistent with this reordering of U. We write the rearranged system matrices as 820 12 Finite Element Displacement Method for Framed Structures ð12:23Þ where U0 contains the unknown displacements, U00 contains the prescribed support movements, P0 E contains the prescribed external nodal loads, and P00 contains the unknown nodal forces (reactions). With this reordering, the system equation takes the following form: ð12:24Þ Expanding the partitioned product results in two matrix equations P0 E ¼ K11 U0 þ K12 U00 þ P0 I P00 E ¼ K21 U0 þ K22 U00 þ P00 I ð12:25Þ We solve the first equation for U0 U0 ¼ K11 1 P0 E P0 I K12 U00 ð12:26Þ Note that the prescribed support movements are converted to equivalent nodal forces. Given U0 , we can evaluate the reactions using the second equation in (12.25). This step involves only matrix multiplication. With this approach, one can deal separately with the nodal loads, initial forces, and support movements. Expanding the right-hand side of (12.26) leads to the following expression for U0. The contributions to U0 from the different loading conditions are: External loading Initial member forces Support movements U0 ¼ K11 1 P0 E U0 ¼ K11 1 P0 I U0 ¼ K11 1 K12 U00 ... Member fixed end actions only: PI 6¼ 0 ... Once the nodal displacements are known, the member end forces are computed using (12.16). 12.6 Introduction of Nodal Supports 821 ð12:27Þ Example 12.6 Support Movement Given: The structure defined in Fig. E12.6a. Consider nodes 1 and 4 to experience support settlements of δ1 and δ4. Fig. E12.6a Determine: The rearranged system matrices Solution: The system matrices are presented in Example 12.4. We place the displacement matrices corresponding to the partially fixed nodes at the bottom. ... The modified form of K consistent with this ordering is generated by moving the first partitioned row and column to the last row and column. The partitioned form of K0 is ... We perform a similar operation on the rows of PE and PI. The corresponding reordered partitioned form for PE and PI are ... Throughout the chapter, matrix computations are carried out using computer software such as MATLAB (29) or MATHCAD (30). Example 12.7: Two-Member Plane Frame—Partially Fixed Given: The frame shown in Fig. E12.7a. Let E, I, and A be constant for both members. Take L ¼ 15 ft, I ¼ 170 in.4, A ¼ 10 in.2, and E ¼ 29,000 ksi. 12.6 Introduction of Nodal Supports 823 Fig. E12.7a Determine: The various matrices for the above structure. (a) Solve for the joint displacements. (b) Determine the member forces and the reactions for the following loading. Solution: Geometric data: The topology and geometric data are summarized below. Member (1) Member (2) α ¼ 0 α ¼ 0.6 ... We determine the individual member matrices using (12.16). These matrices are referred to the global reference frame. ... Assembling the system matrices, the system stiffness matrix takes the following form: ... 12 Finite Element Displacement Method for Framed Structures Introduce displacement constraints: Node 1 is fully fixed, and node 3 is partially fixed, i.e., the rotation θ3 is unknown. We rearrange the rows and columns of K, and the rows of U, PE, and PI. This step leads to the resulting partitioned vectors and matrices listed below. ... 12.6 Introduction of Nodal Supports 827 Introduce the member properties and evaluate the stiffness terms: We evaluate the individual stiffness matrices using the following values for the parameters (L ¼ 15 ft, I ¼ 170 in.4, A ¼ 10 in.2, and E ¼ 29,000 ksi). ... Using the reduced form of K11 we can now evaluate the response for a given loading. Specifying loading: the loading is shown in Fig. E12.7b. Fig. E12.7b Fig. E12.7c Fig. E12.7c The system load vectors corresponding to these loads are: ... Then we obtain the displacements and reactions: ... We determine the member end actions in the global and local frames. ... 12.6 Introduction of Nodal Supports 829 Fig. E12.7d The reactions are listed on Fig. E12.7d. Determine the member end actions: Lastly, we determine the member end actions in the global and local coordinate system using (12.27). Applying these equations for members 1 and 2 leads to the following results: Fixed end actions ... Member forces in the global frame ... Member forces in the local frame ... The local member end actions are listed in Fig. E12.7e. Fig. E12.7e Example 12.8: Two-Member Plane Frame—Fully Fixed Given: The frame shown in Fig. E12.8a. E, I, and A are constant for both members. Take L ¼ 5 m, I ¼ 70(10)6 mm4, A ¼ 6500 mm2, M ¼ 20 kN m, and E ¼ 200 GPa. Fig. E12.8a 12.6 Introduction of Nodal Supports 831 Determine: The reactions Solution: We start with the geometric data. The topological and geometric information is listed below. Geometric data: Member m (1) (2) α 0 −90 Member m (1) (2) n− 1 2 cos α 1 0 n+ 2 3 sin α 0 −1 sin α cos α 1 0 0 −90 cos2 α 1 0 sin2 α 0 1 The system stiffness matrix and displacement vector have the following form: ... We determine the individual member matrices using (12.16). These matrices are referred to the global reference frame. ... Substituting for these matrices, the system stiffness matrix takes the following form using the following values for the parameters I ¼ 5 in, I ¼ 70(10) 6 mm4, A ¼ 6500 mm2, and E ¼ 200 GPa. ... We note that nodes 1 and 3 are fully fixed. We rearrange the rows and columns of K, and the rows of U, PE accordingly. This step leads to the resulting partitioned vectors and matrices listed below (Fig. E12.8b). Fig. E12.8b Nodal forces ... Then we determine the unknown displacements using the following equation: ... The results are shown in Fig. E12.8c. Fig. E12.8c

12.7 Finite Element Displacement Method for Framed Structures

12.7 Specialized Formulation for Beam and Truss Structures

The formulation presented in the previous sections applies for frame type structures which involve both axial and bending actions. Beams and trusses are special cases where only one type of action occurs. Trusses support loads by axial action; beams support transverse loads through bending action. It is relatively easy to specialize the general frame formulation for these limiting cases.

12.7.1 The Steps Involved for Plane Truss Structures

The member end actions for a truss member consist of only an axial force, i.e., there is no shear or moment. Also, each node of a plane truss has only two displacement unknowns; the nodal rotation occurs independent of the translation and has no effect on the member force. We take these simplifications into account by defining "reduced" member force and nodal matrices. The formulation is exactly the same as described in Sects. 12.3–12.6. One only has to work with modified stiffness, rotation, and nodal displacement matrices. Figure 12.7 shows the notation for a truss member in the local frame. Note that now the member matrices are just scalar quantities. ... We consider (12.5) to be the general matrix expression for the member equations for framed structures. Their form follows by deleting the second and third row and column of the matrices listed in (12.6).

Fig. 12.7 End force and displacement—local member system—planar behavior

We need to modify the rotation matrix in a similar way. Now, there are two nodal displacement measures and only one local displacement measure. ... We use this form for R̄g to transform between the local and global frames. Given R̄g, we generate the global form of the member matrices using (12.13). A typical term is ... where k̄l is defined by (12.29). Expanding (12.32) leads to ... and Lastly, we determine the axial force for member m in the local frame. ...

Example 12.9: Statically Determinate Truss

Given: The truss shown in Fig. E12.9a. Fig. E12.9a Determine: The joint displacements, reactions, and member forces using the Displacement Method. A(1) ¼ A(2) ¼ A ¼ 2 in.2, α ¼ 6.5 × 10−6/ F, and E ¼ 29,000 ksi. (a) Due to the loading shown (b) Due to temperature decrease of 40 F for all members. Solution: We start with the geometric data. The topological and geometric information is listed below. Member m (1) (2) ... We determine the individual member matrices using (12.33). ... The system stiffness matrix and displacement vector have the following form: ... We note that node 1 is fully fixed but node 3 is partially fixed. We rearrange the rows and columns of K, and the rows of U. This step leads to the resulting partitioned vectors and matrices listed below. ... Part (a): Loading shown in Fig. E12.9a. The system load vectors corresponding to this loading are ... Given the displacements, one determines the reaction forces using ... Finally, we determine member forces: ... Part (b) ¼ 40 F. We determine the fixed end actions caused by the temperature decrease (Fig. E12.9c and d), and then assemble the PF vectors. Fig. E12.9c Fixed end actions—members 12.7 Specialized Formulation for Beam and Truss Structures Fig. E12.9d Fixed end actions—joints The system load vectors corresponding to these loads are ... We determine the unknown displacement using ... For a determinate truss, the reactions will be zero. Also the member forces will be zero. ...

Nodes 1 and 3 are fully fixed and node 2 is partially fixed. The only unknown displacement is θ2. We rearrange the rows and columns according to the systematic approach described in Sect. 12.6.1. Noting (12.23) and (12.24), the reduced system equation for this example is ... The inverse of K0 11 is ... (i) Loading: We consider the loading shown in Fig. E12.10b. The fixed end actions due to the loading applied to members 1 and 2 are defined in Fig. E12.10d. Fig. E12.10d Substituting for the fixed end actions, the various nodal displacement and load terms reduce to ... (No support movement) ... The unknown displacement is determined with (12.26) specialized for this example ... Then, using (12.27), the reactions are ... Lastly, we compute the member end actions. ... The reactions are listed on Fig. E12.10e. Fig. E12.10e Lastly, we compute the member end actions. ... The results are summarized in Fig. E12.10f. Fig. E12.10f

12.8 Finite Element Displacement Method for Framed Structures (ii) Support Settlement: We consider the support settlement defined in Fig. E12.10g. Noting (12.26), ... at joint 2. The corresponding loading terms are ... We evaluate the reactions with (12.25). In this case, ... Using the given stiffness of ... in the results for Fig. E12.10h. Fig. E12.10g Fig. E12.10h

Example 12.11: Two-Span Beam with a Spring Support at Mid-span 12.7 Specialized Formulation for Beam and Truss Structures Given: The structure shown in Fig. E12.11a. Fig. E12.11a Determine: The displacements at node 2 and the member forces. Take I ¼ 200 in.4 and E ¼ 29,000 ksi. Fig. E12.11b Solution: Nodes 1 and 3 are fully fixed and node 2 is restrained by a linear elastic spring. The unknown displacements at node 2 are the vertical displacement v2 and the rotation θ2. We rearrange the rows and columns of the system matrix consistent with this support condition. The spring is introduced by adding an external nodal force at node 2 whose magnitude is equal to F ¼ −kv2. The minus sign is needed since the spring force acts in the opposite direction to the displacement. We utilize the results for K and PI contained in Example 12.10. The initial and rearranged matrices are ... Noting the matrices listed above, this equation expands to ... Assuming nodes 1 and 3 are fixed, the reduced equations are ... We transfer the term involving v2 over to the right hand side and solve for v2 and θ2. ... When a particular nodal displacement is elastically restrained, we modify the system stiffness matrix K by adding the spring stiffness to the diagonal entry that corresponds to the displacement. We then rearrange the rows and columns to generate K0. Continuing with the computation, the displacements at node 2 are ... Lastly, the reactions are determined with ... The results are listed below. ...

12.8 Three-Dimensional Formulation

In what follows, we extend the planar formulation presented in the previous sections to deal with the case where the loading is three dimensional (3D). The basic approach is the same; one just has to expand the definition of the displacement, end action, and member stiffness matrices. The three-dimensional formulation is much more detailed, and is generally executed using a digital computer. We start by defining the local coordinate system for a prismatic member. We take the X1 direction to coincide with the centroidal axis, and X2 and X3 are the principle inertia directions for the cross section. Figure 12.9 shows this notation. The inertia properties for X2 and X3 are defined as ... There are six displacement measures for the 3D case, three translations and three rotations. The corresponding force measures are defined in a similar manner. We refer these quantities to the local directions and use the notation defined in Fig. 12.10. Using this notation, the 3D versions of the displacement and end action matrices at the end points A and B are ... Fig. 12.9 Local coordinate system for a prismatic member Fig. 12.10 End displacements and end actions in local coordinate system

12.8 Three-Dimensional Formulation We express the end action–end displacement relations in the same form as for the planar case. ... where the k matrices are now of order 6. Their expanded forms are listed below. ... The steps involved in formulating and solving the matrix problem for the planar case are also applicable for the three-dimensional case. The member matrices are transformed to the global reference frame using the following matrix: ... The remaining steps are the same as for the 2D case. One assembles the global equations, introduces the nodal boundary conditions, and solves for the unknown nodal displacements. Simplifications are possible depending on the type of structure. For a space truss there are only axial forces. One can delete the equilibrium equations associated with nodal rotations and moments, and work with the reduced nodal displacement matrix. ... Another case is a plane framed loaded normal to the plane (e.g., a plane grid). One can also delete the equilibrium equations associated with in-plane motion and work with the reduced nodal displacement matrix. Taking the z axis (X3 direction) normal to the plane, the non-zero displacement measures are ...

12.8 Three-Dimensional Formulation Example 12.12: Plane Grid

Given: The plane grid shown in Fig. E12.12a. Determine: The joint displacements, reactions, and member forces. The cross section is square (HSS203.2 × 203.2 × 9.5). Take L ¼ 4 m, P ¼ 30 kN, E ¼ 200 GPa, and G ¼ 77 GPa. Fig. E12.12a Solution: ... Note that there is now a 3 × 3 matrix. ... The unknown displacements are determined using ... Given the displacements, the reactions are ... We note that the displacements are ... Alternative approach: We considered this example in Sect. 10.5 using the force method. ...

Example 12.13: Space Truss

Given: The space truss shown in Fig. E12.13a, b. Determine: The joint displacements, reactions, and member forces. Take A ¼ 1300 mm2, E ¼ 200 GPa. ... The analysis is carried out using MathCAD: ...

12.9 Summary

12.9.1 Objectives

• Formulate the governing equations corresponding to the various steps in the displacement method using matrix notation. • Introduce the concept of a local reference frame for each member and a fixed global reference frame for the structure. • Develop the matrix form of the member force-displacement relations. • Derive expressions for the various member stiffness matrices. • Introduce the concept of a member-node incidence table and show how it is used to assemble the system equilibrium equations. • Develop a procedure for introducing nodal displacement restraints. • Specialize the rigid frame formulation for trusses, multi-span beams, and grids.

12.10 Problems

Problem 12.1 For the rigid frame shown below, use the direct stiffness method to find the joint displacements and reactions for the following conditions: (a) the loading shown, (b) support 1 settles 12 mm. I ¼ 200(10)6 mm4, A ¼ 6500 mm2, E ¼ 200 GPa. Member 2 experiences a uniform temperature change throughout its span. The temperature varies linearly through the depth, from 10 C at the top to 50 C at the bottom. (c) Support 1 settles 12 mm.

Problem 12.2 For the rigid frames shown below, use the direct stiffness method to find the joint displacements, reactions, and member forces. Take I ¼ 160(10)6 mm4, A ¼ 6500 mm2, and E ¼ 200 GPa.

Problem 12.3 For the rigid frame shown, use the direct stiffness method to find the joint displacements, reactions, and member forces. I ¼ 400 in.4, A ¼ 20 in.2, E ¼ 29,000 kip/in.2

Problem 12.4 For the rigid frame shown, use partitioning to determine K0 11, K0 21, P0 E, and P0 I for the loadings shown. I ¼ 400 (10)6 mm4, A ¼ 30000 mm2, and E ¼ 200 GPa.

Problem 12.5 For the truss shown, use the direct stiffness method to determine the joint displacements, reactions, and member forces for (a) the loading shown, (b) a support settlement of 10 mm at joint 4 (c) temperature decrease of 40 C for all members. α ¼ 12 × 10−6/C.

Problem 12.6 For the truss shown, use the direct stiffness method to find the joint displacements, reactions, and member forces due to (a) the loading shown, (b) a temperature decrease of 10 C for all members, (c) a support settlement of δ ¼ 2 mm downward at node 4. A ¼ 1200 mm2 E ¼ 200 GPa α ¼ 12 × 10−6/C h ¼ 3m

Problem 12.7 For the truss shown, determine K0 11 and K0 21. Take A ¼ 2000 mm2 E ¼ 200 GPa.

Problem 12.8 For the truss shown, determine K0 11 and K0 22. A ¼ 3 in.2 E ¼ 29,000 ksi.

Problem 12.9 For the beam shown, use the direct stiffness method to find the joint displacements, reactions, and member forces. I ¼ 300 in.4 E ¼ 29,000 ksi. (a) For the loading shown (b) Support settlement of 12 mm at joint 2.

Problem 12.10 For the beam shown, use partitioning to determine K0 11, K0 21, U00, PI, P0 E and P0 E for the loading and constraint shown. I ¼ 120(10)6 mm4 E ¼ 200 GPa.

Problem 12.11 Investigate the effect of varying the spring stiffness on the behavior (moment and deflected shape) of the structure shown below. Consider a range of values of kv. ...

Problem 12.12 For the beam shown, use partitioning to determine K0 11, K0 21, U00, PI, P0 E and P0 E for the loading and constraint shown.

Problem 12.13 Determine the bending moment distribution and deflection profiles for the following structures. Take I ¼ 300 in.4, Acable ¼ 1 in.2, and E ¼ 29,000 ksi.

Problem 12.14 Determine the bending moment distribution and deflection profile for the following structure.

Problem 12.15 Consider the guyed tower defined in the sketch. The cables have an initial tension of 220 kN. Determine the horizontal displacements at B, C, the change in cable tension, and the bending moment distribution in member ABC. Treat the cables as axial elements. Develop a computer-based scheme to solve this problem. Take Itower ¼ 200(10)6 mm4, Acable ¼ 650 mm2, Atower ¼ 6000 mm2, and the material to be steel.

Problem 12.16 Consider the rigid frame shown below. Investigate how the response changes when I is varied. Use computer software. Vary AI from 2 to 10 in.2 Take I ¼ 100 in.4, and E ¼ 29,000 ksi.

Problem 12.17 (a) Develop a computer code to automate the generation of the member stiffness matrices defined by (12.6). Assume A, E, L, and I, are given. (b) Develop a computer code to carry out the operations defined by (12.13). (c) Develop a computer code to carry out the operations defined by (12.20) and (12.21).

Problem 12.18 For the plane grid shown, use the direct stiffness method to find the joint displacements, reactions, and member forces. The cross section is circular.

Problem 12.19 For the space frame shown, use the direct stiffness method to find displacements at joint B. The load P is applied parallel to member BC. The cross section is circular tube. Take L ¼ 3 m, L2 ¼ 4 m, L3 ¼ 3 m, P ¼ 30 kN, E ¼ 200 GPa, and G ¼ 77 GPa.

Part III Practice of Structural Engineering

The structural engineering process involves identifying possible loading patterns, conceptualizing candidate structural systems, developing idealized physical models, applying the possible loading patterns to these idealized models, using an analysis procedure to determine the peak values of the response variables needed for design detailing, and selecting the design details using an appropriate design code. In this section, we focus on selecting loading patterns, idealizing three-dimensional frame structures, and establishing the peak values of the response variables. Computer-based analysis is used extensively for this phase. Multi-span horizontal structures are discussed in the next chapter. The topics range from girder bridges to arch bridges to cable-stayed bridges. The following chapter presents a strategy for modeling three-dimensional low-rise rigid frame structures subjected to varying loads. Then, the succeeding chapter describes in detail the process followed to establish the critical loading patterns for multistory frames. Finally, the last chapter covers the inelastic response of structures.

13 Multi-span Horizontal Structures

Abstract

In this chapter, we discuss the role of analysis in the structural engineering process for multi-span horizontal structures such as bridges. Typical examples of a girder bridge, an arch bridge, and a cable-stayed bridge are shown in Fig. 13.1. Multi-span girder bridges are actually variable depth horizontal beams. They are used extensively in medium span highway bridge systems. Arch and cable-stayed structures are efficient for spans ranging up to 1000 m. Chapters 9 and 10 dealt with analysis methods for indeterminate structures. Some of the analytical results presented in those chapters are utilized here to estimate critical loading patterns. Most of the analysis effort required in the engineering process is related to determining the peak moment in structures.

Most of the analysis effort required in the engineering process is related to determining the maximum values of bending moment, axial force, and shear corresponding to the typical bridge loadings. Establishing these values for indeterminate structures requires a considerable amount of computational effort. In what follows, we illustrate this computational process for different types of bridge structures such as continuous girders, arch, and cable-stayed schemes using a commercial structural analysis software system.

13.1 The Engineering Process for Girders

The objective of the structural engineering process for girders is to define the physical makeup of the girder, i.e., the location of supports, the material, the shape and dimensions of the cross section, and special crosssection features such as steel reinforcement in the case of a reinforced concrete beam. Given the absolute maximum values of shear and moment at a particular location, the choice of material, and the general shape of the cross section, one can establish the required cross-sectional properties by applying numerical procedures specified by a design code. This phase of the engineering process is called design detailing. We focus here on that aspect of the process associated with the determination of the "maximum" values of shear and moment.

 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_13 873

Fig. 13.1 Bridge structures. (a) Multi-span girder bridge. (b) Arch bridge. (c) Cable Stayed

In general, shear and bending moment result when an external loading is applied to a beam. Throughout the text, we have shown how one can establish the shear and moment distributions corresponding to a given loading. For statically determinate beams, the internal forces depend only on the loading and geometry; they are independent of the cross-sectional properties. However, when the beam is indeterminate, such as a multi-span beam, the internal forces also depend on the 13.2 Influence Lines for Indeterminate Beams Using Müller-Breslau's Principle 875 relative span lengths and cross-sectional properties. In this case, one needs to iterate on the geometry and properties in order to estimate the internal forces. Now, the loading consists of two contributions: dead and live. The dead loading is fixed, i.e., its magnitude and spatial distribution are constant over time. Live loading is, by definition, time varying over the life of the structure. This variability poses a problem when we are trying to establish the maximum values of shear and moment. We need to consider a number of live load positions in order to identify the particular live load location that results in the absolute maximum values of shear and moment. One approach for multi-span beams is based on generating, for all positions of the live load, the absolute maximum value at sections along the span. Plots of global maxima at discrete sections along the span are called force envelopes. An influence line relates a force quantity at a particular point to the location of a loading; whereas a force envelope relates the absolute maximum value of the force quantity along the span. Influence lines provide a way to identify the critical live load positions. In this section, we briefly describe the approach establishes design values.

13.2 Influence Lines for Indeterminate Beams Using Müller-Breslau's Principle

The topic of influence lines for statically determinate beams was introduced in Chap. 3. We include here a discussion of how one can generate influence lines for indeterminate beams using the Müller-Breslau principle [1]. We introduce the principle using the beam structure shown in Fig. 13.2a as an example. Later, in Chap. 15, we apply it to rigid frames. Suppose one wants the influence line for the negative moment at A due to a downward vertical load. According to Müller-Breslau, one works with a modified structure obtained by inserting a moment release at A and applying a negative moment at A. The resulting structure is shown in Fig. 13.2b. The deflected shapes of the modified structure due to a unit load applied at an arbitrary point, and a unit negative moment at A, are plotted in Fig. 13.2c, d. Since the beam is indeterminate, a compatibility requires the net relative rotation at A to vanish. Then θAΔ þ θAA MA ¼ 0 ... Since δxA is at an arbitrary point, it follows that the deflected shape of the modified structure due to a unit moment is a scaled version of the influence line for MA. The actual sense of MA is determined by comparing the direction of the displacement with the direction of the applied load. In this example, the positive sense of MA needs to be reversed when the load is applied in the positive sense. The displacement profile corresponding to (13.1) is shown in Fig. 13.3, which is based on the sign convention for moment, i.e., positive when compression on the upper fiber. 876 13 Multi-span Horizontal Structures

Fig. 13.2 Application of Müller-Breslau principle. (a) Example structure. (b) Negative moment. (c) Deflection due to load. (d) Deflection due to moment

Fig. 13.3 Loading zones for moment at A. (a) Negative moment at A. (b) Positive moment at A

13.2 Influence Lines for Indeterminate Beams Using Müller-Breslau's Principle 877 We repeat this process to establish the influence line for the maximum positive moment at D, the center of span AC. The sequence of steps is illustrated in Fig. 13.4. Figure 13.5 defines the loading zones for positive and negative moment. Summarizing the discussion presented above, the process of applying the Müller-Breslau principle for a redundant force quantity involves the following steps: 1. Modify the actual structure by removing the restraint corresponding to the force quantity of interest. 2. Apply a unit value of the force quantity at the release and determine the deflected shape. 3. This deflected shape is a scaled version of the influence line. If the quantity is an upward force, the positive zones include those regions where the deflected shape is upward. Since it is relatively easy to sketch deflected shapes, the Müller-Breslau principle provides a convenient way to identify the loading patterns with minimum effort.

Fig. 13.4 Modified structure for positive moment at D

Fig. 13.5 Loading zones for moment at D

Example 13.1 Application of Müller-Breslau Principle

Given: The four-span beam shown in Fig. E13.1a. Fig. E13.1a Determine: The influence lines for the upward vertical reaction at support 3 (R3), the negative moment at support 3 (M3), the positive moment at section 1-1 (M1–1), and shear at section 1-1 (V1–1). Use these influence lines to determine the critical loading patterns for a uniformly distributed load that produce the maximum values of R3, M3, M1–1, and V1–1. Solution: The deflected shapes and influence lines are plotted in Figs. E13.1b–e. Fig. E13.1b Influence line for R3 Fig. E13.1c Influence line for M3 Fig. E13.1d Influence line for M1–1 Fig. E13.1e Influence line for V1–1 The loading patterns for the absolute maximum values of force parameters are shown in Figs. E13.1f, E13.1g, E13.1h, and E13.1i. Fig. E13.1f Loading patterns for absolute maximum R3 Fig. E13.1g Loading patterns for absolute maximum M1–1 Fig. E13.1h Loading patterns for absolute maximum M3 Fig. E13.1i Loading patterns for absolute maximum V1–1

13.3 Engineering Issues for Multi-span Girder Bridges

13.3.1 Geometric Configurations

The superstructure of a typical highway girder bridge consists of longitudinal girders which support a concrete deck. The girders may be fabricated from either steel or concrete. The substructure is composed of piers and abutments which transmit the loads to the ground. The foundation type depends on the soil conditions at the site. Bearings are employed to connect the girders to the substructure. Bridge spans are classified as either short, medium, or long according to the span length. Typical categories are: Short: less than 125 ft (38 m); Medium: 125–400 ft (38–120 m); Long: over 400 ft (120 m). Typical highway girder bridges are constructed using an arrangement of simply supported beams. However, this choice requires additional bearings and introduces discontinuities in the deck slab at the interior supports. The discontinuity creates a serious problem since it provides a pathway for moisture and leads to corrosion of the deck and girders. Continuous girders eliminate the need for interior joints in the deck slab and also eliminates some bearings. It is the preferred structural scheme for new bridges. Typical span arrangements are shown in Fig. 13.6. The current trend is to use a constant girder cross-section throughout the span. 13.3 Engineering Issues for Multi-span Girder Bridges 881

Fig. 13.6 Span arrangements for multispan beams. (a) Two spans continuous. (b) Two spans simply supported. (c) Three spans continuous. (d) Three spans simply supported

Historically, girder bridges were configured as a collection of single spans. This scheme is illustrated in Fig. 13.7a. In order to deal with longer interior spans, the cantilever scheme shown in Fig. 13.7b was introduced. Both schemes involve discontinuities in the girder/deck which provide pathways for moisture to flow into the girder and the substructure. To eliminate the interior discontinuities, the obvious option is to use a continuous girder as shown in Fig. 13.7c. As illustrated in Chap. 9, continuous beams are more efficient structurally, i.e., the peak values of the bending moment are reduced in comparison to the corresponding forces for the simply supported case. Therefore, the required cross section tends to be lighter. Even when a continuous girder is used, there is still remains the problem of the discontinuities at the end supports (abutments). The problem of end discontinuities can be eliminated by using the scheme shown in Fig. 13.8. The abutments walls are supported on flexible piles that are rigidly connected to the deck/girder. This concept is called an "integral abutment bridge." Since the abutment is rigidly attached to the deck/girder, a temperature change of the deck produces a longitudinal displacement of the abutment wall. In order to minimize the effect of the longitudinal loading on the piles, flexible piles and loose granular backfill is placed behind the wall. The longitudinal displacement due to temperature varies linearly with the span length, and consequently, the maximum span length is governed by the seasonal temperature change. We generate an idealized model for the bridge structure by replacing the action of the soil and piles with equivalent springs [2]. Figure 13.9a illustrates this approach. An estimate of the effect of support stiffness is obtained using the model shown in Fig. 13.9b.

Fig. 13.7 Multi-span bridge configurations. (a) Simple spans. (b) Cantilever spans. (c) Continuous spans

13.3.2 Choice of Span Lengths

Given some overall crossing length, one needs to decide on the number and relative magnitude of the spans to be used to achieve the crossing. We utilize here some of the analytical results generated in Chaps. 9 and 10 pertaining to continuous beams. Figure 13.10 shows the maximum moment values for spans with increasing number of spans. Note that there is a significant reduction in peak moment as the number of spans is increased, for a given overall length. Note also that the bending moment distribution for constant I is independent of the value of I. In general, for constant I, the bending moment distribution depends on the relative span lengths. For the symmetrical case shown in Fig. 13.11, the analytical solution for the negative moment at an interior support has the form ... We express L1 and L2 in terms of the total span. The variation of M with α is plotted in Fig. 13.12. ... Fig. 13.8 Integral abutment bridge 883 Fig. 13.9 (a) Three-span integral abutment bridge in Orange, Massachusetts. (b) Elevation—three-span integral abutment bridge (c) Global model. (d) Simplified model

Fig. 13.10 Moment diagrams for multispan beams. (a) Two spans. (b) Three spans. (c) Four spans

Fig. 13.11 Bending moment distribution—three-span symmetrical scheme

Fig. 13.12 Variation of moment with α

The more common case is where α is between 1/3 and 1/2. When spans L1 and L2 are chosen, one uses the uniformly distributed loading and computes the moment ... Taking L2 ¼ L1 corresponds to α ¼ 1/3.

13.3.3 Live Loads for Multi-span Highway Bridge Girders: Moment Envelopes

The live loading for a highway bridge is generally assumed to consist of two components: a uniformly distributed lane loading intended to simulate small vehicles, such as cars, and a set of concentrated loads that represent heavy vehicles, such as trucks. The critical action for a heavy vehicle truck loading is a set of concentrated loads. The load magnitude and spacing vary with the design code; the code used here is AASHTO. The moment values depend on the location of the loads on the span. When the beam is statically determinate, it is possible to develop an analytical solution for the peak moment. This approach is described in Chap. 3. However, when the beam is indeterminate, one needs to resort to a numerical procedure. The procedure for generating moment envelopes at specific points along the span is shown in Fig. 13.12. In practice, one first discretizes the spans and then positions the load at the individual discrete points. Assuming there are n discrete points, one needs to carry out n analyses. This results in n bending moment distributions. At each discrete point, one determines the maximum positive and negative values from the set of n values generated by the n analyses. Finally, we construct a plot showing the "maximum" values of moment at each discrete point. This plot allows one to readily identify the absolute "maximum" moment by scanning over the plot. Given the "maximum" moment envelopes for each span, we generate the moment envelope. Working with a refined span discretization provides detailed 886 13 Multi-span Horizontal Structures information. We usually take 10 points per span ... Fig. 13.12 Truck and lane loading. (a) Truck load. (b) Span discretization. (c) Three-axle truck. (c) Multiple truck positions Given the discretization, one needs to generate the moment envelope for the span discretization shown in Fig. 13.12c. We discuss next how one establishes the magnitudes of the concentrated loads.

13.3.3.2 Transverse Distribution of Truck Load to Stringers

Figure 13.13 shows typical slab stringer highway bridge cross sections. The roadway is supported by a reinforced concrete slab, which rests on a set of longitudinal beams, usually called stringers. The stringers may be either steel sections or concrete sections. In order to determine the loading on the stringer, we position the truck such that one set of wheels is directly on the stringer. Figure 13.14 illustrates this case. Note that P is the axle load. Fig. 13.13 Typical slabstringer bridge cross sections. (a) Steel girders. (b) Concrete T beams. (c) Precast concrete beams Fig. 13.14 Transverse position of vehicle wheel loads 888 13 Multi-span Horizontal Structures We assume the slab acts as a simply supported beam spanning between the stringers. The axle distribution factor is defined as DF ... The load on the stringer is ... This approximation is conservative. Then we have for a stringer "A" ... One needs also to account for the effect of the dynamic impact of the load. When the load is applied rapidly as the vehicle travels onto the bridge, a measure of the loading duration is the ratio of span length to vehicle velocity. The dynamic amplification of a suddenly applied load is equal to 2. In the case of a moving truck, the effect is much less. This effect is usually accounted for by increasing the live load. An impact magnification factor (I) of 30 % is commonly used. With this notation, the load applied to the stringer is given by Pi design ¼ ... where Pi is the axle load.

13.3.3.3 Uniform Live Load

Small vehicles are modeled as a uniformly loading applied selectively to individual spans. The purpose of this loading is to simulate the case where a set of passenger cars is stalled in a lane on one or more spans. The load pattern for the uniform loading is defined by the influence lines for the moments at the midpoint of the interior spans and the supports. For a three-span structure, the critical loading patterns for moments are shown in Fig. 13.15. Load cases 1 and 2 produce the peak positive moment at mid-spans. Cases 3 and 4 produce the peak negative moment at the interior supports. Fig. 13.15 Lane load cases—loading patterns. (a) Influence lines for positive moments at mid-spans. (b) Influence lines for negative moments at the supports. (c) Maximum positive moment at mid-spans (cases 1 and 2). (d) Max negative moments at the supports (cases 3 and 4) 889 890 13 Multi-span Horizontal Structures Given the loading patterns, one generates the moment distribution for each pattern and then establishes the maximum values of the positive and negative moments. These results define the discrete moment envelope corresponding to the lane loading for this three-span structure. Four separate analyses (cases 1–4) are required to construct the discrete moment envelope for the lane loading.

13.3.4 Loading Due to Support Settlements

In addition to the gravity loading associated with the weight of the deck and vehicles, one also needs to consider the moments induced in the structure due to support settlement. This calculation is relatively straightforward. The analytical solutions for two- and three-span symmetrical beams are generated in Examples 10.2 and 10.5. We list these results in Fig. 13.16 for convenience. Note that the peak moment varies as 1/L2. Therefore support settlement is more significant for short spans vs. long spans.

13.3 Engineering Issues for Multi-span Girder Bridges Fig. 13.16 Moments due to support settlements. (a) Two-span case for vA. (b) Two-span case for vB. (c) Three-span case for vA. (d) Three-span case for vB 891 892 Example 13.2 Effect of Span Length on Support Settlement Moments Given: The two-span beam shown in Fig. E13.2a. The beam properties are E ¼ 200 GPa and I ¼ 9000(10)6 mm4. Fig. E13.2a Determine: The bending moment distribution due to support settlement of 25 mm at supports A and B. Consider the following cases: (a) L ¼ 10 m, (b) L ¼ 20 m Solution: The bending moment distributions are plotted in Figs. E13.2b and E13.2c. These results demonstrate that the effect of support settlement is more critical for the shorter span [case (a)]. Fig. E13.2b Case results Case a: L ¼ 10 m Fig. E13.2c Case results Case b: L ¼ 20 m

13.4 Case Studies

13.4.1 Case Study I: A Three-Span Continuous Girder Bridge

We illustrate the process of establishing design values using an actual bridge as a case study. The bridge is a three-span structure with spans of 80 ft, 110 ft, and 80 ft, with an overall length of 270 ft. The superstructure consists of an 8-in. thick concrete slab acting in composite with four lines of steel girders spaced at 8.67 ft on center. The girders have constant cross-section. The deck carries two traffic lanes, continuous over the interior supports; the bearings are either hinge or roller supports. Figures 13.17 and 13.18 show the makeup of the bridge system and the details of the cross section. The bridge is modeled using an equivalent section equal to approximately one-fourth of the cross section of the bridge (one girder plus a 8.67 ft slab). Figure 13.18b defines the model used for this analysis. Our objectives are 1. To determine the moment envelopes for truck and lane loading corresponding to a live uniform lane loading of 0.64 kip/ft and the standard truck load. Fig. 13.17 Girder bridge system 894 13 Multi-span Horizontal Structures Fig. 13.18 (a) Cross section—bridge deck. (b) Cross section of single composite beam Fig. 13.19 Truck load 2. To establish the absolute peak values (positive and negative) for moment due to dead loading of 2.1 kip/ft, lane loading, and design truck loading. 3. To determine the moments due to: 1 in. settlement of support A; 1 in. settlement of support B. Loading patterns: We adopt the lane loading and the truck loading shown in Figs. 13.20 and 13.21. We discretize the individual spans into ten segments, as shown in Fig. 13.22. A computer software system is used to generate the solutions for each load case. Fig. 13.20 Uniform lane loading patterns Fig. 13.21 Truck loading Fig. 13.22 Span discretization for truck loading, ... The lane load patterns are defined in Fig. 13.21 and are analyzed separately; the maximum values... Fig. 13.23. We generate envelopes ... Note that it is necessary to consider all load cases. Although the structure is symmetrical, the results are not symmetrical. Fig. 13.24 Uniform lane load envelopes Fig. 13.25 Truck loading envelopes

Table 13.1 Discrete envelope values: Span I (kip ft) ... Table 13.2 Discrete envelope values: Span II ... Table 13.3 Discrete envelope values: Span III ...

13.4 Case Studies 13.4.2 Case Study II: Two-Hinged Arch Bridge We consider the two-hinged arch bridge shown in Fig. 13.26. The structure is a typical two-hinged parabolic arch; the span is 16.5 m, ... We model the roadway as a continuous longitudinal beam supported at 10 ft intervals by ... Fig. 13.27 Two-hinged arch bridge Fig. 13.28 Parabolic-arch geometry and loading-idealized model Fig. 13.29 Force envelopes—Truck loading—100 straight segments. (a) Moment envelope. (b) Axial envelope 13.4 Case Studies 901 Fig. 13.30 Force envelopes—Truck loading—10 straight segments. (a) Bending moment envelope. (b) Axial envelope ... The force envelopes for the truck loading are plotted in Figs. 13.29 and 13.30. Figure 13.29 is obtained by dividing the arch into 100 straight segments having a constant projection, Δx, of 0.3 m. Figure 13.30 is generated by subdividing the arch into 10 straight segments having a constant projection, Δx, of 3 m. The force envelope plots are very similar. The ratio of the bending moment to axial force is the same order as the absolute values ... Note that the ten segment model produces quite reasonable results. 902 13 Multi-span Horizontal Structures

13.4.3 Case Study III: Three-Span Parabolic Arch Response—Truck Loading

We consider next the three-span arch shown in Fig. 13.31. The span lengths, discretizations, and the truck loading are the same as for case study II. It is of interest to compare the peak values of the force envelopes for the two different structural models. The structure consists of straight segments having a constant horizontal projection of 1 ft. A computer software package was used to generate the corresponding force envelopes which are plotted in Fig. 13.32. Comparing the moment envelopes for the girder and arch, we note that arch system has lower peak moment values. However, the arch system has axial forces so that the arch must be designed for combined bending and axial action. The arch and girder cross sections are designed to resist the moments in the girder system, just pure bending. Fig. 13.31 Idealized model—three-span arch. (a) Axial envelope—threespan arch 13.4 a Case Studies 903 Fig. 13.32 Force envelopes—three-span arch. (a) Geometry and loading. (b) Moment envelope Fig. 13.33 Force envelopes—three-span girder.

13.4.4 Case Study IV: Cable-Stayed Bridge

This case study concerns the cable-stayed bridge concept, a type of structure that requires some special modeling strategies and exhibits a completely different behavioral pattern than girder and arch-type structures. A typical configuration is shown in Fig. 13.34. The terms "harp" and "fan" refer to the positioning of the cables on the tower. A modified fan arrangement is usually adopted to avoid congestion on the tower. Fig. 13.34 Typical cable-stayed scheme 904 13 Multi-span Horizontal Structures Fig. 13.35 Idealized cable-stayed structure Of particular interest is the load path for vertical loading applied to the girder. Without the cables, the girder carries the load by bending action throughout the total span. Since the moment in the girder is a function of the square of the span length, this span is not feasible for long spans. The effect of the cables is to provide a set of vertical supports to the girder, thus reducing the moment in the girder. In what follows, we illustrate this effect using the idealized structure shown in Fig. 13.35. We suppose the girder is continuous, and the cable layout is symmetrical (equally spaced on the vertical tower). There are seven pairs of symmetric cables. Each cable has a different crosssectional area. The girder is hinged at the tower, and the ends of the girder are simply supported. We model the cables as axial members that are hinged at their ends to the tower and the girder. In this way, they function as axial elements and transmit the gravity loading applied to the girder up to the tower. The net effect is to reduce the bending moment in the girder. Starting with nodes at the supports and the cable-girder intersection points, one may also discretize the girder between the cable nodes to obtain more refined displacement and moment profiles. The objective of the analysis is to establish the forces in the cables, the cable tensions and the girder moments. The cable forces depend on the stiffness of the cables and the girder. The girder is assumed to be quite flexible, and therefore the cables have to be designed with areas that are proportional to their tension forces, the cable is mostly loaded in bending. In a typical design, one could begin by assuming an initial value of the cable area. The displacement of a cable node ... Fig. 13.36 Tributary area for cable We estimate the cable force, ... Cable C1 C2 C3 C4 C5 C6 C7 θ 19.6 22.6 26.5 32 39.8 51.3 68.2 1 sin θ 2.9 1.9 1.6 1.3 1.1 Acable (mm2) 1305 1130 957 827 696 566 480 13.4 Case Studies 905 Fig. 13.36 Tributary area for cable We estimate the initial cable areas listed below. ... The girder cross-sectional area is taken as 120,000 mm2. Note that the bending moment in the girder is similar to the moment for the girder by assuming the bending moment diagram is similar to the distribution for a uniformly loaded beam with multiple supports. Fig. 13.37 shows the forces and displacement profile corresponding to Igirder ¼ 420(10)6 mm4 and the following set of cable areas for cables 1–7, respectively (1305, 1130, 957, 827, 696, 566, and 480 mm2). Given these estimated properties, one analyzes the structure and iterates on the properties until the design requirements are satisfied. The girder cross-sectional area is taken as 120,000 mm2. Note that the girder is similar to that of a multi-span uniformly loaded beam. We also point out that the response is sensitive to the girder cross-sectional area since there is significant compression in the girder. We point out that the axial force increases towards the tower. The bending moment and displacement profiles for the girder agree with this approximation. The peak value occurs for the outermost cable which has the largest length and smallest angle. A suggested peak value for cable area under live load is L/800, which for this geometry translates to 300 mm. We can decrease the deflection by increasing the area of the outer cables. Assuming an individual cable acts as a single vertical spring subjected to the loading w(ΔL), and requiring the displacement to be equal to wall leads to the following estimate for the cable area Ac ¼ wΔLĽLc vall E sin θ2 where Lc is the cable length. Holding the girder properties constant, we use this approximate expression to increase the cable areas to (14,000, 11,500, 8500, 5000, 3000, 3000, 2000 mm2) and repeat the analysis. The displacement profile of the girder for this case is plotted in Fig. 13.38 and also summarized in the table listed below. Note that the displacement is sensitive to the cable area and the angle of inclination; the cable tension is governed primarily by strength. This case study illustrates the 906 13 Multi-span

Horizontal Structures Fig. 13.37 Force and displacement profiles. (a) Geometry and loading. (b) Moment in girder. (c) Axial forces in cables and girder. (d) Displacement profile of girder Fig. 13.38 Final displacement profile of girder 13.5 Summary 907 role of computer simulation in executing the design of cable-stayed structures. One refines the design through iteration. This example also illustrates how cable-stayed structures carry the load primarily through axial action in the cables. The girder functions mainly to transmit the deck loading to the cables, i.e., the bending of the girder is localized between the cable support points. Cable C1 C2 C3 C4 C5 C6 C7 13.5 Acable (mm2) 1305 1130 957 827 696 566 480 Tension (kN) 914 810 665 567 467 387 315 v# (mm) 2382 1864 1341 940 630 428 287 Acable (mm2) 14,000 11,500 8500 5000 3000 3000 3000 Tension (kN) 998 761 675 566 467 386 386 v# (mm) 297 217 186 176 157 86 48 Summary 13.5.1 Objectives • To present Mu¨ller-Breslau's principle and illustrate how it is used to establish loading patterns that produce the maximum value of a force quantity at a particular point on a structure. • To describe a procedure for determining the load on an individual stringer due to an axle load applied to the deck of a slab-stringer bridge system. • To describe and illustrate a computer-based procedure for generating force envelopes for indeterminate horizontal structures subjected to a set of concentrated loads. • To illustrate the different behavioral patterns for multi-span girder, arch, and cable-stayed systems. 13.5.2 Key Facts and Concepts • Mu¨ller-Breslau's principle is used to establish influence lines for indeterminate structures. One works with a structure generated by removing the constraint provided by the force quantity. The deflected shape of the structure due to a unit value of the force quantity is a scaled version of the influence line. • The moment envelope for a horizontal structure is generated by positioning the loading at discrete points on the longitudinal axes, tabulating the moment at each discrete point for all the loading cases, and selecting the largest positive and negative values. A computer-based procedure is used for this task. • Support settlement can produce bending moments which are significant for short span bridges. 908 13.6 13 Multi-span Horizontal Structures Problems Problem 13.1 1 1 2 3 1 1 4 1 5 1 10 ft 30 ft 40 ft 40 ft 30 ft (a) Using Mu¨ller-Breslau's principle, sketch the influence lines for the vertical upward reaction at support 3 (R3), the negative moment at support 2 (M2), and the negative moment at section 1–1 (M1–1) (b) Use a software package to determine: (i) The maximum values of R3, M2, and M1–1 caused by a uniformly distributed dead load of 2 kip/ft. (ii) The maximum value of M2 caused by a uniformly distributed live load of 1 kip/ft. Problem 13.2 Consider the single span bridge shown below. Using the analytical procedure described in Sect. 13.2.1, determine the absolute maximum value of moment produced by the truck loading defined below passes over the span. Repeat the analysis using computer software. Problem 13.3 Consider the two-span bridge shown below. Use computer software to determine the global moment envelopes for the lane load and truck loadings defined below. Lane load: w ¼ 10 kN/m uniform Truck load: (a) L1 ¼ L2 ¼ 30 m, EI is constant (b) L1 ¼ L2 ¼ 30 m, EI is constant 13.6 Problems 909 (c) L1 ¼ L2 ¼ 30 m, I2 ¼ 1.5 I1 (d) L1 ¼ 30 m, L2 ¼ 40 m, EI is constant (e) L1 ¼ 30 m, L2 ¼ 40 m, I2 ¼ 1.5 I1 (f) L1 ¼ 30 m, L2 ¼ 40 m, I2 ¼ 1.5 I1 (d) Compare the global moment envelopes for the structure shown below with the envelopes generated in part (a). Is there any effect of varying I? Problem 13.4 Consider the multi-span bridge shown below. Suppose the bridge is expected to experience a temperature change of ΔT over its entire length. Where would you place a hinge support: at A or at B? Determine the end moment corresponding to your choice of support location. 910 13 Multi-span Horizontal Structures Problem 13.5 Most design codes limit the deflection due to live loading to some fraction of length, say L/α, where α is on the order of 500. Generate the global "deflection" envelope for the multi-span beam and truck loading shown below. Take E ¼ 29,000 ksi and I ¼ 60,000 in.4 Problem 13.6 Investigate convergence of the internal forces for the parabolic arch shown as the discretization is refined. Take the interval as 2.4, 1.2, and 0.6 m. Problem 13.7 Suppose a uniform loading is applied to span ABC. Investigate how the response changes as x varies from L/2 to L. Take h ¼ L/2, A ¼ 50 in.2, AC ¼ 2 in.2, w ¼ 1 kip/ft, I ¼ 4 = (2x/ L). 12. 13.6 Problems 911 Problem 13.8 Determine the structural response (forces and displacements) of the idealized tied arch shown below under a uniformly distributed gravity load of 30 kN/m. Assume: Arch A ¼ 26,000 mm2, IArch ¼ 160(10)6 mm4, Ahanger ¼ 2(10)6 mm2 Note: roadway girder and arch are pinned together at points A and B. An actual structure is shown below. Problem 13.9 Determine the distribution of internal forces and displacements for the cable-stayed structure shown below. Member AB acts as cantilever beam supporting equally spaced cables. Self-weight of members AB and BC is 16 and 8 kN/m, respectively. Assume ACable ¼ 50,000 mm2 and E ¼ 200 GPa. Consider the following cases: (a) IAB ¼ IBC ¼ 40(10)9 mm4 (b) IAB ¼ 4IBC IBC ¼ 40(10)9 mm4 (c) Uniform live load of 2 kN/m applied on member BC in addition to self-weight. IAB ¼ 4IBC IBC ¼ 40(10)9 mm4 912 13 Multi-span Horizontal Structures Problem 13.10 Consider the structural concept created by Santiago Calatrava is shown below. This bridge is located in Seville, Spain. This work has been released into the public domain by its author, Consorcio Turismo Sevilla. This applies worldwide. The image was accessed in March 2012 from Puente del Alamillo.jpg. Assume the properties of the structural components and loading. Problem 13.10 Consider the symmetrical cable structure shown below. Determine a set of cable areas (C1–C10) such that the vertical displacement is less than 375 mm under a uniformly distributed live load of 10 kN/m. Assume Igirder ¼ 400(10)6 mm4, Agirder ¼ 120(10)3 mm2. Take the allowable stress as 700 MPa. References 1. Wilbur JB, Noris CH. Elementary structural analysis. New York: McGraw-Hill; 1948. 2. Faraji S, Ting J, Crovo DS, Ernst H. Nonlinear analysis of integral bridges: finite element model. ASCE J Geotech Geoenviron Eng. 2001;127(5):454–61. Lateral Load Issues for Buildings 14 Abstract Buildings are complex physical systems. Structural Engineers deal with this complexity by creating idealized structural models that define the key structural components, the overall makeup of the building structure, and the loadings that it needs to withstand. The information provided by the idealized model allows one to apply analysis and design methods directly to the model and then extrapolate the results to the actual building. We begin the chapter with a description of the various types of building systems and the associated structural components. In general, a building consists of plane frame structures which are interconnected by floor systems. We describe approaches for establishing the lateral loads due to wind and earthquake excitation. These loads are evaluated at each floor level and then distributed to the individual plane frames using the concepts of center of mass and center of twist. At this point, one has the appropriate lateral loading to apply to the plane frames. The topic of loading on building frames is discussed further in the next chapter where we also consider gravity loads acting on the floor systems. 14.1 Types of Multistory Building Systems The majority of the activities in Structural Engineering are concerned with the design of Structural Systems for buildings. Approximately 95 % of the building inventory consists of buildings having less than ten stories. Buildings of this type are classified as low-rise buildings. Figure 14.1 illustrates the typical makeup of a low-rise building. The primary structural components are beams, columns, and floor plates. Members are usually arranged in an orthogonal pattern to form a three-dimensional framework. Plate-type elements span between the beams to form the flooring system. We visualize the three-dimensional (3D) framework to be composed of plane frames which are connected by floor plates. This interpretation allows us to analyze the individual plane frames rather than the complete 3D structure. # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3 14 915 916 14 Lateral Load Issues for Buildings Fig. 14.1 Typical makeup of a structural system for a low-rise building Fig. 14.2 (a) Steel braced frame with bracing. (b) Concrete frame with shear wall Bracing Shear wall Frames are designated according to how the beams and columns are interconnected at their end points. When the members are rigidly connected so that no relative rotation can occur, end moments develop under loading and the frame is said to be "rigid." Rigid frames may employ either Steel or Concrete construction. The opposite case is when the beams are pinned to the columns. No end moments are developed and the frame behaves similar to a truss. Some form of bracing is needed since a rectangular pinned frame is unstable under lateral load. These structures are called "braced" frames. Figure 14.2a shows a typical braced frame structure. The bracing consists of sets of diagonal elements placed within certain bays and extending over the height of the structure. This system is designed to carry all the lateral loading. Note that at least two orthogonal bracing systems are needed to ensure stability under an arbitrary lateral loading. Braced frames are constructed using steel components. 14.2 Treatment of Lateral Loading 917 Depending on the magnitude of the lateral loading, lateral stiffness systems may also be incorporated in rigid frames to carry a fraction of the lateral load. For concrete rigid frames, the stiffening is achieved by incorporating shear walls located either within or on the exterior of the building and extending over the entire height. Figure 14.2b illustrates this scheme. These walls function as cantilever beams and provide additional lateral restraint. For steel rigid frames, the stiffening system may be either a concrete shear wall or a diagonal steel member scheme. 14.2 Treatment of Lateral Loading Lateral loading is due to either wind or earthquake acting on the building. These actions may occur in an arbitrary direction. For rectangular buildings, such as shown in Fig. 14.3, the directions are usually taken normal to the faces. One determines the component of the resultant force for each direction. Figure 14.4 illustrates this approach. The resultant force is distributed to the individual floors, and then each floor load is distributed to the nodes on the floor. This process leads to a set of nodal forces acting on the individual frames. We express the force acting at floor j of frame i as Px frame i floor j ¼ f j Rx ð14:1Þ How one establishes fij is discussed in Sect. 14.3. Fig. 14.3 Rectangular building Fig. 14.4 Resultant force components Ry frame n frame i frame 2 frame 1 floor j z Y floor 2 floor 1 X f1j Rx 918 14 Lateral Load Issues for Buildings 14.2.1 Wind Loading We suppose the wind acts in the x direction, as shown in Fig. 14.5a. The normal pressure varies in the vertical direction according to a power law (e.g., p ~ z1/7). We approximate the distribution by a set of step functions centered at the floor levels and generate the force for each floor by integrating over the tributary area associated with the floor. This process is illustrated in Fig. 14.5b, c. The individual floor forces are given by hz z z i hz z i ip1 i i1 ip1 i1 Pi ¼ p2i P i Þ P B ð14:2Þ 2 2 2 i ¼ 1, 2, . . . , n a n n-1 Pi 3 2 Y 1 X b B z n n–1 i Wind zi 3 2 1 Y X Fig. 14.5 Lateral floor forces due to wind pressure. (a) Wind in X direction. (b) Wind on Y–Z face. (c) Floor loads due to wind load on Y–Z face 14.2 Treatment of Lateral Loading 919 c p(z) n n n-1 i i 2 2 1 1 Pn n-1 Pn-1 Pi i P2 2 1 P1 Fig. 14.5 (continued) This computation is repeated for wind acting in the Y direction. It remains to distribute the loads acting at the floor levels of the facade to nodes of the individual plane frames. The final result is a set of lateral nodal loads for each plane frame. The underlying strategy for this approach is based on analyzing plane frames vs. a threedimensional system. This approach works when the structure is composed of parallel plane frames which produce an orthogonal pattern of columns and beams. If the structural geometry is irregular, one has to analyze the full 3D structural system. In this case, one subdivides the fac¸ade area into panels centered on the structural nodes contained in the fac¸ade and generates the force for structural node j with Pj ¼ p z j Aj ð14:3Þ where Aj is the tributary area for structural node j. 920 14 Lateral Load Issues for Buildings 14.2.2 Earthquake Loading Earthquake loading is generated by an earthquake passing through the site. An earthquake is the result of slippage between adjacent tectonic plates which releases energy in the form of pressure waves that produce ground motion having both horizontal and vertical ground motion. For civil structures, in seismically active regions, the horizontal motion produces the most critical lateral loading since the design of civil structures is usually controlled by vertical gravity loading. Data on earthquake ground motion is continuously collected and distributed by the US Geological Survey National Earthquake Information Center [1]. Figure 14.6 contains a typical plot of ground acceleration vs. time for the 1994 Northridge California earthquake. The information of interest is the peak ground acceleration, denoted as pga, with respect to g, the acceleration due to gravity. In this case, the pga is equal to 0.6g. We point out that seismic loading is cyclic, of varying amplitude, and of short duration, on the order of 20–30 s for a typical earthquake. Seismic loading is discussed in Sect. 1.3.6. We briefly review the important features of seismic loading here and then describe how one uses this information to generate the lateral loading for a building. The horizontal forces produced by horizontal ground motion require the incorporation of lateral bracing systems. Structures located in high seismic activity regions, such as Japan, Greece, and the Western parts of the USA, are required to meet more extreme performance standards, and the design is usually carried out by firms that specialize in seismic design. Figure 14.7 illustrates how a typical low-rise rigid frame building responds to horizontal ground motion. The floor slabs act like rigid plates and displace horizontally with respect to the ground due to bending of the columns. Since there are no external loads applied to the floors, the deformation has to be due to the inertia forces associated with the floor masses. The magnitude of these forces depends on the floor masses and the floor accelerations. The lateral displacement profile is assumed to be a linear function of Z as indicated in Fig. 14.8. u ðtÞ ¼ uground ð14:4Þ Fig. 14.6 Ground acceleration time history— Northridge (1994) California Earthquake Ü u Z ð H Þ 14.2 Treatment of Lateral Loading 921 Fig. 14.7 Seismic response of low-rise rigid frame Fig. 14.8 Lateral displacement profile with respect to ground where u(H) is the relative displacement of the top floor with respect to the ground, and Zi is the vertical coordinate of floor i. This assumption leads to the following expression for the total acceleration of a typical floor: Z i d2 uð H Þ P afloor i ¼ ag ðt Þ þ H dt2 ð14:6Þ where ag(t) is the ground acceleration time history. Applying Newton's law, the force required to accelerate floor i, assuming it moves as a rigid body and the lateral displacement profile is linear, is Wi Z i d2 uð H Þ ag ðt Þ þ P floor i ¼ g H dt2 ð14:5Þ This force is provided by the shear forces in the columns adjacent to the floor. Given an earthquake ground motion time history, one applies this floor loading to the structure and determines the structural response. The solution for the acceleration at the top floor is expressed as [2]: d2 uð H Þ ¼ Γ ag ðt Þ þ θðt Þ ð14:6Þ dt2 where θ(t) depends on the earthquake ground motion and the structural design; Γ is a dimensionless parameter that depends on the distribution of the floor masses X N Wi Z i ¼ W i Z i Γ ¼ ¼ 1 N X ¼ 1 W i Z 2i ð14:7Þ Substituting for the top floor acceleration, the inertia force expands to Wi Z i Wi Z i ag ðt Þ þ Γ θðt Þ P floor i ¼ g H g H ð14:8Þ The peak values of ag(t) and θ(t) are not generally occur at the same time. Also the magnitude of Γ is of order one and the maximum value of θ(t) is usually larger than ag max . Therefore, the peak force at floor i is approximated as: Zi Wi Sa P floor i ¼ ð14:9Þ H g where Sa is defined as the maximum absolute value of θ(t). In this form, an maximum acceleration that an equivalent single degree of freedom system experiences when subjected to the earthquake. Summing up the floor forces leads to the resultant force which is also equal to the maximum base shear. X N X N Wi Z i Sa V base ¼ ð14:10Þ Pfloor i ¼ X N Z g W Ž P i¼1 i¼1 Finally, we express the force for floor i in terms of V|base : ! W Z i i Pfloor i ¼ X N V base W ð14:11Þ The spectral acceleration measure, Sa, depends on the ground motion time history ag(t) and the period of the structure, T. A simple approximation for T for a low-rise building is T N s 10 ð14:12Þ where N is the number of stories. Values of Sa vs. the structural period, T, have been compiled by various agencies, such as the US Geological Survey's National Earthquake Information Center [1] for a range of earthquakes, and used to construct design plots such as illustrated below in Fig. 14.9. One estimates T and determines Sa with this plot. The limiting values for the plot, such as SDS and SD1, depend on SMS and SM1 which are defined for a particular site and seismic design code [3]. Values of SMS, SM1, and TL are listed on the USGS Web site, usgs/gov/hazards: SM1 is usually taken as the spectral acceleration for 5 % damping and 1 s period (i.e., T1 ¼ 1 s): SMS is the spectral acceleration for 5 % damping and 0.2 s period and TL is the long transition period. The worst case scenario is for the structural period to be between 0.2TS and TS. When T T S , the seismic load is significantly less than the load corresponding to the region 0.2TS < T < TS. 14.2 Treatment of Lateral Loading 923 Fig. 14.9 Peak acceleration vs. structural period [3] Example 14.1 Given: The three-story building shown in Figs. E14.1a and E14.1b. Assume the building is subjected to an earthquake in the North–South direction. Take the spectral acceleration as Sa ¼ 0.4 g W3 = 900 kip 12 ft W2 = 1000 kip 12 ft W1 = 1000 kip 12 ft 100 ft Fig. E14.1a Elevation N-S 100 ft S N Pi 100 ft Fig. E14.1b Typical floor plan Determine: The base shear and earthquake forces on the individual floors. Solution: We use (14.10). The base shear is given by X 3 2 Sa f1000ð12Þ þ 1000ð24Þ þ 900ð36Þg2 Sa i¼1 Z i W o ¼ W i ¼ 0:4 ¼ 1368 kip V base ¼ X 3 1000ð12Þ2 þ 1000ð24Þ2 þ 900ð36Þ2 g W ðZ Þ2 g i¼1 i i¼1 924 14 Lateral Load Issues for Buildings Then, applying (14.11), we obtain the individual floor loads. X 3 i¼1 W i Z i ¼ f0ð12Þ1000Þ þ 24ð1000Þ þ 36ð900Þg ¼ 68, 400 12ð1000Þ V base ¼ 0:175ð1368Þ ¼ 239:4 kip 68, 400 24ð1000Þ V base ¼ 0:351ð1368Þ ¼ 480:2 kip P2 ¼ 68, 400 36ð900Þ V base ¼ 0:474ð1368Þ ¼ 648:4 kip P3 ¼ 68, 400 P1 ¼ 239.4 kip 1 Example 14.2 Given: The three-story building shown in Figs. 14.2a and E14.2b. The floor weights are indicated. There is also an additional weight located on the top floor. W = 1000 kip W3 = 900 kip 12 ft W2 = 1000 kip 12 ft W1 = 1000 kip 12 ft 100 ft Fig. E14.2a Elevation 100 ft S N Pi 100 ft Fig. E14.2b Typical floor plan 14.2 Treatment of Lateral Loading 925 Determine: The earthquake floor forces for a North–South earthquake with intensity Sa ¼ 0.4 g. Solution: The computations are organized in the following table. Floor 1 2 3 Roof Zi 12 24 36 36 Wi 1000 1000 900 1000 WiZi (103) 12 24 32.4 36 104.4 103 Wi(Zi)2(103) 144 596 1050 1296 3086 103 ð104:4ð1000Þ2 ¼ 1412:7 kip V base ¼ ð0:4Þ 3086ð1000Þ P1 ¼ 12, 000 ð1412:7Þ ¼ 162:3 kip 104:4ð1000Þ P2 ¼ 24ð1000Þ ð1412:7Þ ¼ 324:8 kip 104:4ð1000Þ P3 ¼ ð32:4 þ 36Þð1000Þ ð1412:7Þ ¼ 925:6 kip 104:4ð1000Þ Note that the shear in the top story is increased considerably due to the additional mass on the third floor. 14.3 Building Response Under Lateral Loads Up to this point, we have discussed how one generates the lateral loads acting at the floor levels. These loads are resisted by the frames which support the floor. In this section, we develop a methodology for distributing a floor load to the frames which support the floor. We model the building as a set of rigid floors supported by columns and braces shown in Fig. 14.10. When subjected to horizontal loading, the floor plates displace horizontally, resulting in bending of the columns and shearing deformation in the braces. The horizontal load is resisted by the shear forces developed in the columns and braces. We know from the examples studied in Chaps. 9 and 10 that stiffness attracts force. Therefore, one should expect that the distribution of floor load to the supporting elements, i.e., columns and braces, will depend on the relative stiffness of these elements. 926 14 Lateral Load Issues for Buildings Fig. 14.10 (a) One-story braced frame. (b) Shear spring model for brace 14 Lateral Load Issues for Buildings L a B kB kC kD kA H V, u b V, u V, u V u k V=ku Plan Elevation Fig. 14.11 Plan view L FCx kc B/2 PX w u o KD FDy n kB FBy B M kA FAx L/2 Py 14.3.1 Center of Twist: One-Story Frame We consider first the one-story braced frame structure shown in Fig. 14.10a. The braces located at the midpoints of the sides provide the resistance to horizontal load. We represent the braces as simple shear springs. Figure 14.10b illustrates this modeling strategy. Each brace provides a force which acts in the plane of the wall that contains the brace. We locate the origin of the X-Y coordinate system at the geometric center of the floor and assume the floor is rigid. Therefore, the floor will experience translation (u, v) and rotation ω about the origin. The free body diagram for the floor is shown in Fig. 14.11. 14.3 Building Response Under Lateral Loads 927 Noting the free body diagram shown, the equilibrium equations are X þ! Fx ¼ 0 Px FAx FCx ¼ 0 X "þ Fy ¼ 0 Py FBy FDy ¼ 0 X M 0 ¼ 0 L B M 0 þ FDy FBy þ FCx FAx ¼ 0 2 2 ð14:12Þ The brace forces are related to the displacements by B FAx ¼ kA u þ ω 2 B FCx ¼ kC u ω 2 L FBy ¼ kB v þ ω 2 L FDy ¼ kD v ω 2 ð14:13Þ Substituting for the forces in (14.13) leads to B Px ¼ u ðkA þ kC Þ þ ð kA kC Þω 2 L Py ¼ v ðkB þ kD Þ þ ð kD kB Þω 2 B L B L2 M ¼ ðkA kC Þu þ ðkD kB Þv þ ðkA þ kC Þ þ ðkB þ kD Þ ω 2 2 4 4 ð14:15Þ The displacements depends on the relative stiffness of the braces. If they all have equal stiffness, the equations uncouple, i.e. Px u¼ kA þ kC v¼ Py kB þ kD ð14:16Þ M ω¼ ðkA þ kC Þ B2 ðkB þ kD Þ L2 þ 4 4 Note that floor rotation occurs only when M0 is not equal to zero. Therefore, rotation will occur only when Px or Py are applied at the geometric center. If we take kA = kC or kB ≠ kD, the response is coupled in that u, v, and ω depend on the loads. Example 14.3 Given: The floor shown in Fig. E14.3a. The brace stiffnesses are shown in Fig. E14.3a. Ly = k D = kB = k A = 4k 100 ft kc = k A 10 k 10 m Fig. E14.3a Plan View Determine: The response due to Px. Solution: We note that the braces at A, B, D have equal stiffnesses, and the brace at C is twice as stiff as the others. For convenience, we show these values as just k and 2k. We set Py ¼ M ¼ 0 in (14.15). 928 14 Lateral Load Issues for Buildings Px ¼ 3kðu þ 5ω Þ 6 5ðkÞ þ Px=ð6k 20 ¼ 500 k u¼ Mþ k ω þ 5ðk ω Þ 0 Solving these equations, we obtain 259 78k ¼ ω P x ð¼ 1/5k 8 ¼ kB þ kC ¼ 2kω ¼ 5.2 8 25 5 8 Finally, the brace forces are FAx ¼ kA u þ ¼ ¼ Px P x Px 0:7 70 7 0 FCx ¼ kC u − 2 ¼ 70 14 L L 5 FBy ¼ kB v þ ω ¼ 2 70 14 FDy ¼ kD v − ω ¼ 2 70 14 To avoid rotation, which is undesirable, one needs to modify either the stiffness at A or C. Taking kA ¼ kC ¼ k, the response is P2 ¼ 2kω ¼ 0. Px ¼ 2ku FAx ¼ FCx ¼ Px 2 The formulation described above can be generalized to deal with an arbitrary number of braces or shear walls oriented in either the X or Y direction. We shift to the rotation shown in Fig. 14.12 to Fig. 14.12 Notation 14.3 Building Response Under Lateral Loads 929 identify the various braces. Each brace is characterized by a stiffness magnitude (k) and the perpendicular distance from the tangent to the origin. Assuming the floor is rigid, the tangential motion for the X oriented braces due to a rigid body motion of the origin is ðu þ þ ! ui ¼ u yi ω ð14:17Þ Similarly, the Y motion for brace j is given by v þ xj ω ð14:18Þ vj ¼ v ! þ The individual forces are þ ! Fxi ¼ kxi uyi ω þ Fyj ¼ kyj v þ xj ω ð14:19Þ Summing forces and moments with respect to the origin leads to the following equations for the floor X Px ¼ kxi u yi ω ð X Py ¼ kyj v þ xj ω X X M 0 ¼ kxi u yi ω yi þ kyj v þ xj ω xj Þ ð14:20Þ We define the following terms: X kxi ¼ Kx X Kyj ¼ Ky X X kyj X kxi yi ¼ Kxy X kyj xi ¼ Kyx X X kxi yi2 þ kyj x2j ¼ Kz ð14:20Þ With this notation, (14.19) takes the following form: Px ¼ K x u Kxy ω Py ¼ K y v þ Kyx ω ð14:21Þ M0 ¼ Kyx u þ Kxy v þ Kzω Equation (14.21) applies for an arbitrary choice of origin. Note that forces applied at the origin will produce rotation when either Kxy ≠ 0 or Kyx ≠ 0. If stiffness distribution is symmetrical, these terms vanish. Rotation of the floor is a torsional mode of response, which introduces an undesirable anti-symmetric deformation in the perimeter facades. Therefore, one needs to avoid this mode. Another approach is to shift the origin to some other point in the floor. Obviously, the most desirable point corresponds to Kxz ¼ Kyz ¼ 0. Consider the floor geometry shown in Fig. 14.13. Point o denotes the initial origin and C some arbitrary point in the floor. We locate a new set of axes at C and express the forces in terms of the coordinates with respect to C. C 930 14 Lateral Load Issues for Buildings Fig. 14.13 Floor geometry o ¼ yi ωi þ vC yj ¼ xi ω ¼ xC þ ω ω ¼ ωC C ω ¼ ω ð14:22Þ ð14:23Þ Then, the moment equation becomes Px ¼ Kx uC Kxy ωC Py ¼ Ky vC þ Kyx ωC M 0 ¼ Kxy uC þ Kyx vC þ Kz ωC ð14:24Þ We choose point C such that the coupling terms vanish. The new forces are X kxi yi ¼ Kxy X ¼ 0 X kyj xj ¼ 0 yC ¼ Kyx / Kx ... (14.25) These conditions define the coordinates of point C. Substituting for x0 and y0 using 0 x ¼ xC þ x 0 y ¼ yC þ y and solving for xC and yC leads to X X kyj xj kxi yi X X xC ¼ kyj yC ¼ kxi ð14:26Þ The equilibrium equations referred to these new axes simplify to Px ¼ Kx uC Py ¼ Ky vC MC ¼ M0 þ Py xC Px yC ¼ Kz ωC ð14:27Þ where KC ¼ KC ¼ X kxi ðyi yC Þ2 þ X kyj xj xC 2 ð14:28Þ Equations (14.27) and (14.28) show that the resultant of the "resisting" forces acts at the center of twist. External forces applied at the center of twist produce only translation; an external moment applied to the floor produces twist only. We point out that the location of the center of twist depends on the distribution of the components located in the story. The location of the center of twist changes when either the position or the magnitude of the stiffness components is changed. Example 14.4 Given: The floor plan shown in Fig. E14.4a. The two shear walls are orthogonal and are located at the X and Y axes. Fig. E14.4a Determine: The center of twist. Solution: The center of stiffness lies on an axis of symmetry. In this case, there are two axes of symmetry and therefore the center of twist is at the origin. Example 14.5 Given: The stiffness distribution shown in Fig. E14.5a. 932 14 Lateral Load Issues for Buildings Fig. E14.5a Determine: The center of twist. Solution: The stiffness distributions are symmetrical with respect to the X and Y axes. Therefore, XCT ¼ YCT ¼ 0: Example 14.6 Given: The stiffness distribution shown in Fig. E14.6a. Fig. E14.6a Determine: The center of twist (CT). Solution: The center of twist (CT) lies on the X-axis because the stiffness is symmetrical with respect to the X-axis. Summing moments about the origin leads to X xi kyi ¼ ð3kb þ 2kbÞ ¼ 5kb ¼ k xCT ¼ 2 4k* ¼ X Z 4k* kyi Example 14.7 Given: The stiffness distribution shown in Fig. E14.7a. 14.3 Building Response Under Lateral Loads 933 Fig. E14.7a Determine: The center of twist. Solution: The floor will experience rotation as well as translation since there is a net moment with respect to the center of twist. Summing moments about the origin, the moment measures using (14.27), we obtain the result below. X kx ¼ 2k* Kxy ¼ 2k kb Ky ¼ kx k* Note, Px Px ¼ X kx 2k* Py yi ¼ 4k* u¼ Py b Px ¼ b2 P b Py ðb ¼ k ... Example 14.8 Given: The stiffness distribution shown in Fig. E14.8a. The general configuration of the floor is shown. Determine: The location of the center of twist for the following cases: (a) Fig. E14.8b k3 k1 k3 k1 k2 k3 14 Lateral Load Issues for Buildings Example 14.8 Determine: The stiffness distribution shown in Fig. E14.8a. Case (a) k4 k3 k1 k2 Case (b) k4 k2 k3 k1 Case (c) k4 > k3 > k1 > k2 Solution: The problem can be viewed as being equivalent to finding the centroid of a set of areas, with areas replaced by stiffness. One can use qualitative reasoning to estimate the location of the center of twist. Case (a) k1 ¼ k2 ¼ k3 ¼ k4 14.3 Building Response Under Lateral Loads 935 Case (b) k4 ¼ k3 k1 > k2 Case (c) k4 > k3 > k1 > k2 Case (d) k4 > k3 k1 > k2 We consider next the single inclined brace shown in Fig. 14.15. Introducing displacements u and v produces a longitudinal force F equal to ku0 . Projecting F on the x and y axes leads to h i Fx ¼ F cos θ ¼ k u0 cos θ ¼ k u cos θ þ v sin θ cos θ h i Fy ¼ F sin θ ¼ k u sin θ cos θ þ v sin θ sin θ ð14:28Þ We write these equations as Fx ¼ kxx u þ kxy v Fy ¼ kxy u þ kyy v ð14:29Þ where kxx ¼ k cos2 θ kxy ¼ k sin θ cos θ kyy ¼ k sin2 θ ð14:30Þ When θ is on the y axis, the brace is a pure y brace. Noting the stiffness, we see that there is no coupling. 936 14 Lateral Load Issues for Buildings Fig. 14.15 Inclined brace Summing over the number of braces, the resultants are given by Fy ¼ u = 0 ¼ v sin θ cos θ ð14:31Þ ... We see that when θ ¼ 0 or 90, these equations reduce to (14.27). The inclined braces produce coupling terms in the equilibrium equations. Now it is necessary to specify summary moment about O. Working first with the X component of F, the translation of point o: 2 Kxx Kxy XX ... Summing forces in the direction of the coordinates, the following pair of equations for x* and y* , the coordinates of the center of twist: X X kyi sin θi cos θ y* Kxy þ X kxi ... Note that when the stiffness elements are parallel to either x or y, these equations reduce to (14.26). The center of twist CT is located at (x*, y*). Example 14.9 Given: The stiffness distribution shown in Fig. E14.9a. Two of the braces are inclined with respect to the x-axis. Y k1 k2 h/2 k2 X O h/2 45° 45° k1 b/2 b/2 Fig. E14.9a Determine: The center of twist. Solution: Evaluating (14.31) leads to 1 K cc ¼ k1 cos 0 ¼ 2k1 ... 2 X yi ki sin θ cos θ ¼ 0 X 2 Then, bk2 y* ¼ 2k1þk2 ... ð14:32Þ 14.3.2 Center of Mass: One-Story Frame We consider next the distribution of mass. The center of mass is of interest for a one-story frame that is subjected to an earthquake. When the frame is subjected to a ground motion, the floors experience accelerations, and inertia forces are generated due to the acceleration of the masses located on the floor. In order to study the equilibrium of the floor, we need to establish the magnitude and location of the resultant inertia force acting on the floor. This resultant acts at the center of mass. In what follows, we describe a procedure for locating this resultant. Figure 14.16 presents a typical plan view of a floor. We locate the origin at some arbitrary point in the floor, and suppose that there are masses located at discrete points in the floor. The center of mass is a particular point in the floor defined by the coordinates x and y, where X xi mi xCM ¼ X mi X yi mi yCM ¼ X mi ð14:32Þ Example 14.10 Given: The floor mass layout shown in Fig. E14.10a. 14.3 Building Response Under Lateral Loads 939 Determine: The center of mass. Solution: The center of mass is on the y-axis. In general, if the mass distribution is symmetric with respect to an axis, the center of mass lies on that axis of symmetry. We determine the y coordinate by summing moments about the x-axis (Fig. E14.10b). Example 14.11 Given: The floor mass layout shown in Fig. E14.11a. Determine: The center of mass. Solution: Summing moments leads to 940 14 Lateral Load Issues for Buildings Example 14.12 Given: The floor mass layout shown in Fig. E14.12a. Determine: The center of mass. Solution: We sum moments about the x and y axes and obtain the results. 14.3.3 One-Story Frame: General Response We have shown that there are two key points in the floor, the center of twist and the center of mass. We now proceed to examine the response of the floor subjected to a load with quantities referred to the center of twist. Noting (14.27), the response of the floor for an arbitrary static loading is (Fig. 14.17) Px K x Py vC ¼ Ky MCT ωC ¼ KC ð14:33Þ Note that twist occurs only when there is an external moment with respect to the center of twist; forces applied at the center of twist produce only translation. 14.3 Building Response Under Lateral Loads 941 Fig. 14.17 Forces acting at the center of twist Fig. 14.18 Forces acting at the center of mass (a) Displacements. (b) Forces When the loading is dynamic, one needs to include the inertia forces. In this case, it is more convenient to place the origin at the center of mass and work with force and displacement quantities referred to the axes centered at the center of mass. Figure 14.18a illustrates this choice. The resultant forces act at the center of mass and produce a moment about the center of twist. The displacements of the two centers are related by ... 942 14 Lateral Load Issues for Buildings The equilibrium equations referred to the center of mass have the following form: Px ¼ m€uCM þ Kx uCM ... ð14:35Þ shows that the motion is coupled when the center of twist does not coincide with the center of mass. The center of mass is usually fixed by the mass distribution on the floor and one usually does not have any flexibility in shifting masses. Therefore, the most effective strategy is to adjust the location of the braces in the story below the floor such that the centers of mass and twist coincide, i.e. To have xCT = xCM and yCT = yCM. Example 14.13 Given: The floor shown in Fig. E14.13a. Fig. E14.13a Determine: The magnitude of the stiffness elements k1 and k2 such that the centers of mass and twist coincide. Solution: First, we locate the center of mass. X mi ¼ 20 m X xi mi ¼ 20(m) ... Example 14.13 Building Response Under Lateral Loads 943 yi mi ¼ 100m ... Requiring the center of twist to coincide with the center of mass, we determine k1 and k2 by requiring ... The steps are Step 1: X kx ¼ k1 þ 20k2 ... 14.3.4 Multistory Response A typical multistory frame is connected to the adjacent floors by stiffness elements such as columns, shear walls, and braces. When the floors displace, inter-story deformation due to the relative movement of adjacent floors is produced in the stiffness elements. Figure 14.19 illustrates this mode of behavior. Floors i and i + 1 experience lateral displacements which produce shear deformations in the braces ... The inter-story shear forces act on both floor i and i+1. We know from Newton's law of action and reaction that these forces are equal and opposite. In order to express these resistance forces in terms of the displacement measures for the floors, we need to specify a common reference frame for all the floors. We suppose the floors translate and rotate with respect to their individual centers of twist. We determine the inter-story displacement measures for the story by subtracting ... These conditions refer to the center of twist for story i ... 944 14 Lateral Load Issues for Buildings Fig. 14.19 Forces due to inter-story deformation Fig. 14.20 Forces acting on floor i defined in Fig. 14.20. The forces acting on floor i+1 are opposite to those due to story i. Now we consider the equilibrium of floor i ... 14.3 Building Response Under Lateral Loads 945 Fig. 14.21 Inertia forces for floor i ... Figure 14.21 shows the inertia forces. We require floor i to be in equilibrium. Summing forces with respect to the origin at O leads to the following equilibrium equations: ... In regions where the seismic loading is high, such as California, one needs to consider dynamic response. In this case, the goal in seismic design is to have the center of mass and center of twist coincide throughout the height of the structure. The formulation obtained above can be interpreted as a "shear beam" formulation for a building system in the sense that the assumptions we introduced concerning the behavior of a floor are similar to those for a beam subjected to shearing and torsional action. These assumptions are applicable for low-rise buildings, where the aspect ratio, defined as the ratio of height to width, is of order 1. Most buildings are in this category. For tall buildings and for those structures having flexible floors, one creates idealized models consisting of 3D frame structures composed of beams, shear walls, and floor plates. These models generally involve a large number of variables and require computer-based analysis methods to generate solutions. The advantage of simple models is that one can reason about behavior through examination of analytical solutions. Both approaches are necessary and each has a role. 946 14 Lateral Load Issues for Buildings 14.3.5 Matrix Formulation: Shear Beam Model In what follows, we introduce matrix notation and express the equations derived in the previous section in a form similar to the equations for a member system that are presented in Chap. 12. We number the floor and stories consecutively, and work with the common X–Y–Z reference frame shown in Fig. 14.22. The displacement and force quantities for floor i are expressed as matrix ... We define the inter-story displacements at the center of twist for story i ... Using (14.41) and (14.43), (14.45) expands to ... 14.3 Building Response Under Lateral Loads 947 where Kj depends on the stiffness properties of story j. ... We need to transform these forces from the center of twist to the origin of the common reference frame. ... This operation involves the transpose of TCT. ... Continuing with the properties of the center of twist namely, Kxx, Kyy, KC, xCT, yCT, and Ko are determined ... The displacements at the center of mass are related to the acceleration matrix by ... 948 14 Lateral Load Issues for Buildings Finally, summing forces and assembling ... We suppose there are N floors and express the complete set of N equations as a single matrix equation, € þ KU P ¼ mU ð14:54Þ We assemble m and K in partitioned form (N rows and N columns). The entries follow from (14.53). ... Note that this assembly process is similar to the procedure that we followed in Chap. 12 to assemble the system matrices for a member system. The example listed below illustrates the steps for a three-story structure. Example 14.14 Given: The three-story structure shown in Fig. E14.14a. Determine the system equilibrium equations. Solution: ... The non-zero entries in the system load, mass, and stiffness matrices ... 14.4 Response of Symmetrical Buildings 949 14.4 Response of Symmetrical Buildings We consider next the response of symmetrical buildings. When the floors are symmetrical, the center of mass and center of twist both lie on the Z-axis. In this case, the equations uncouple and one can deal with the translation in the X direction separately from the translation in the Y direction. We consider a load applied in the X direction. The floors displace in the X direction and develop shears due to shearing action in the braces. A typical frame is modeled as a set of discrete masses supported by shear springs. Figure 14.24 shows this idealization. The shear spring stiffness for a story in a frame is determined by summing the contribution of the columns contained in the story. Using the approximate method for estimating lateral stiffness for frames developed in Chap. 11, the equivalent shear stiffness for a story in a frame is estimated as kstory i ¼ 12E X I 12E X I c sum ... where r is the ratio of relative stiffness of the girders and columns. Example 14.15 ... the story stiffness is ... The shear stiffness factors for the shear walls are computed ... Fig. 14.25 Shear stiffness elements. (a) Steel brace. (b) Concrete shear wall kstory i ¼ kcol story i þ kbrace story i ð14:57Þ The shear stiffness factors for braces shown in Fig. 14.25 are kshearwall ¼ kbrace ¼ h Gbt 2AE ð ð14:58Þ ð14:59Þ The complete building system is represented as a set of frames in parallel linked through the "rigid" floor slab. Figure 14.26 illustrates this idealization. At each story level, all frames experience the same lateral displacement. It follows that the story shear force in a particular frame is proportional to the ratio of the frame story stiffness to the global story stiffness. ... Noting Fig. 14.26, the global shear force in story i is equal to the sum of the loads acting on the floors above the particular floor. For example, 14.4 Response of Symmetrical Buildings 951 ... One first evaluates these global shear forces and then determines the individual frame story shears with (14.60). Suppose the ratio of story stiffness is constant for all stories in frame j. ... This result is useful since it allows one to reason in a qualitative way about how global floor loads are distributed into the frames. For example, suppose that there are n frames having equal stiffness. Then, each frame carries (1/n) of the total load. Example 14.15 Given: The symmetrical rigid frame structure shown in Fig. E14.15a. Assume the frame properties are constant throughout the building height and the structure is uniformly loaded. (a) The columns in frames 2 and 3 are twice as stiff as the columns in frames 1 and 4 and the floor slab is rigid. (b) Assume equal frame stiffnesses and rigid floor slab. (c) Assume equal frame stiffnesses and a flexible floor slab. 952 14 Lateral Load Issues for Buildings ... Determine: The distribution of the total lateral load to the individual frames. Solution: Part (a): A typical floor is shown in Fig. E14.15b. The equivalent shear stiffness factors are defined as k* and 2k*. The stiffness distribution is symmetrical. Fig. E14.15b Typical floor The total story stiffness is X kj ¼ k* þ 2k* þ 2k* þ k* ¼ 6k* According to (14.60), the fraction of the total story shear carried by an individual frame is equal to the ratio of the frame stiffness to the total story stiffness. Then, For frames 1 and 4 ... For frames 2 and 3 ... 14.4 Response of Symmetrical Buildings 953 ... In this case, the interior frames carry twice as much load as the exterior frames. Part (b): If the floor slab is rigid and the frame stiffnesses are equal, the frame loads are equal. Part (c): Suppose one can consider the floor slab to be flexible. Fig. E14.15c Flexible floor slab. Then we need to consider the tributary areas for each frame. ... The floor load is distributed to the frames according to the tributary area. We divide the fac¸ade into area segments and associate these segmental areas with the frames adjacent to the areas as illustrated in Fig. E14.15d. ... This breakdown is shown in Fig. E14.15e. This distribution is based on the assumption that the frames act independently, i.e., the floor slab is flexible. Example 14.16 Given: The five-story symmetrical rigid frame structure shown in Figs. E14.16a and E14.16b. Assume the building can be subjected to an earthquake in either the North-South or East-West directions. Take the spectral acceleration as Sa ¼ 0.15g. Consider all the columns to be the same size and all the girders to be the same size. Assume IB ¼ 4IC. Fig. E14.16a Elevation 14.4 Response of Symmetrical Buildings 955 Fig. E14.16b Typical floor Determine: The maximum moments in the columns (a) for rigid floors and (b) for flexible floors. Solution: We use (14.10). The base shear is given by ... Then, applying (14.11), we obtain the individual floor loads (Fig. E14.16c). ... 956 14 Lateral Load Issues for Buildings Fig. E14.16c Earthquake floor loads It remains to distribute the floor loads to the frames. Since the structure is symmetrical, we need to consider only one direction, say the N–S direction. Fig. E14.16d Floor load distribution Part a: When the floor slab is rigid, and the frame stiffnesses are equal, the floor load is distributed uniformly to the frames (Fig. E14.16e). ... 14.4 Response of Symmetrical Buildings 957 Fig. E14.16e Frame load—rigid floors Part b: When the floor slab is flexible, the loads are proportioned to their tributary floor areas. ... Fig. E14.16f Frame load—flexible floors Part c: We apply the stiffness method described in Chap. 11 to estimate the maximum moments in the columns. 958 14 Lateral Load Issues for Buildings ... Fig. E14.16g Maximum column moments—rigid floor 14.4 Response of Symmetrical Buildings 959 Fig. E14.16h Maximum column moments—flexible floors Example 14.17 Given: The one-story frame shown in Figs. E14.17a, E14.17b, E14.17c. Assume the cross sections are equal. Fig. E14.17a Plan 960 14 Lateral Load Issues for Buildings Fig. E14.17b Elevation—section 1-1 Fig. E14.17c Elevation—section 2-2 Determine: (a) The center of mass. (b) The center of twist. (c) The revised stiffness required in lines B-B and 2-2 so that the center of twist coincides with the center of mass. (d) The rotation and translation of the floor due to the earthquake load acting at the center of mass. Solution: Part (a): The center of mass ... Part (b): The center of twist ... Using the stiffness relations listed in Chap. 11 (11.11), (11.12), the relevant stiffness factors are ... 14.4 Response of Symmetrical Buildings 961 (b) The center of twist ... Let (EIc/h3) ... Then, 3EI CE ... When steel members are used, the usual approach is to form the floor by first installing joists, then overlaying steel decking, and lastly casting a thin layer of concrete. Loading applied to the floor is transmitted through the decking to the joists and eventually to the beams supporting the joists. For the geometry shown in Fig. 15.6, beams and all carry essentially all the loads applied to the floor and are designed for a uniformly distributed load. The loads on beams and are associated with the small tributary areas between the beam and the adjacent joists. Depending on the joist spacing, one may ... The loading patterns are shown in Fig. 15.7 are listed below. 15.2 Approaches for Live Load Patterns 963 ... Fig. 15.6 Typical beam and joist span. Fig. 15.7 One-way action beam loading for uniform floor loading q. (a) Large joist spacing. (b) Small joist spacing ... 15.3 Live Load Patterns for Frame Structures Gravity type loading is usually the dominant loading for low-rise multistory frames. It consists of both dead and live loading. Given a multistory frame, the first step is to establish the critical loading patterns for the individual members. Once the loading patterns are established, one can carry out an approximate analysis to generate peak force values which are used for the initial design. From then on, one iterates on member properties using a computer-based analysis method. In this section, we describe how Mu¨ller-Breslau's Principle can be employed to establish loading patterns for gravity loads. We also describe some approximate techniques for estimating the peak positive and negative moments in beams. Consider the frame shown in Fig. 15.8. Suppose we want the loading pattern for live load which produces the maximum positive moment at A. Using the Mu¨ller-Breslau Principle, one applies a downward load to those spans where the beam deflection is upward to produce the maximum positive moment at A. The corresponding loading pattern is shown in Fig. 15.10. Referring back to Fig. 15.8, we establish the loading pattern for negative moment at B by inserting a moment release at B and applying a negative moment. In this case, there are two possible deflected shapes depending upon whether one assumes the inflection points are in the columns Fig. 15.9 Loading pattern example A A A B 15.3 Live Load Patterns for Frame Structures 981 Fig. 15.10 Loading pattern for maximum positive moment at A A Fig. 15.11 Deflection pattern for negative moment at B—inflection points in columns + + + + B + + + Fig. 15.12 Deflection pattern for negative moment at B—inflection points in beams + + + + B + of the beams. Given the approximate nature of this analysis, one generally adopts the pattern corresponding to the shape of negative moment ... The Mu¨ller-Breslau Principle is a very useful tool for generating a qualitative first estimate of the loading patterns for the structure. One can then calculate specific response using an analysis software system. 982 15 Vertical Loads on Multistory Buildings Fig. 15.13 Loading pattern for maximum moment at B—inflection points in columns B Example 15.1 Given: The rigid steel frame shown in Fig. E15.1a. Determine: (a) Critical loading patterns for gravity live loading using Mu¨ller-Breslau's Principle that produces the peak values of maximum moments at mid-spans and end points of the beams. (b) Use a computer software package to determine the maximum moment corresponding to the critical patterns of loading and compare the results to the results for a uniform loading on all members. Consider all the girders to be the same size and all the columns to be the same size. Assume L ¼ 6 m, L2 ¼ 9 m, h ¼ 4 m, w ¼ 10 kN/m, and IG ¼ 3IC Fig. E15.1a Rigid frame example 15.3 Live Load Patterns for Frame Structures 983 Solution: Part (a): The process followed to determine the critical loading patterns for bending moment in the beams is described in the following steps. Step 1: Positive Moment at mid-span of the beams. There are four loading patterns for positive moment at the midpoint of the beams. They are listed in Fig. E15.1b. Fig. E15.1b Positive moment loading patterns Step 2: Negative moment at the end point of the beams There are 16 loading patterns for negative moment. Typical patterns are shown in Fig. E15.1c. One carries out analyses for the 15 different loading patterns, and then represents the results by a discrete moment envelope. Figure E15.1d shows the final results. 984 Fig. E15.1c Typical loading patterns for negative moment 15 Vertical Loads on Multistory Buildings 15.3 Live Load Patterns for Frame Structures 985 Fig. E15.1d Discrete moment envelope for uniformly distributed load Part (b): The moment results for uniform loading are plotted in Fig. E15.1e. We note that the uniform loading results underestimate the peak values (30 % for positive moment and 11 % for negative moment). However, since the uniform loading case is easy to implement and provides reasonable results, it frequently is used to generate a first estimate. Fig. E15.1e Moment diagram for uniform loading 986 15 Vertical Loads on Multistory Buildings Example 15.2 Given: The three-story symmetrical rigid frame building shown in Figs. E15.2a, E15.2b, and E15.2c. Assume the building is subjected to a uniform gravity load and an earthquake in the north–south direction. Consider the floor load to be transmitted to all sides (two-way action). Assume all the columns and beams are made of steel. Fig. E15.2a 3D model. Fig. E15.2b Typical floor plan 15.3 Live Load Patterns for Frame Structures 987 Fig. E15.2c Typical frame—specified floor loads on a typical frame Determine: The shear, moment, and axial force distributions for a typical interior bay and the lateral displacement of the floors using the following properties. Properties are specified for two cases: The second case corresponds to doubling the column inertias for case one Iz = 445,146, 750 mm4 I shape beams Iy = y22, 798,170 mm4 Ix = 1,135, 750 mm4 z A = 12,320 mm2 Iz = 148,520,925 mm4 case (1) Ix = 233,390, 025 mm4 A = 10, 677 mm2 Hollow square columns Iz = Iy = 309,106,575 mm4 case (2) y z Ix = 486, 749, 250 mm4 A = 15,867 mm2 Solution: The floor loading is due to the uniform load applied to the floor slab. Using the concept of tributary areas, we convert this loading to line loadings on the perimeter floor beams. Note that the N–S and E–W loading are identical because of the symmetry. W floor total ¼ 24ð2000Þ kN 2000 ¼ 2:74 kN=m w2 ¼ 2:40 ¼ 12:35 kN=m w3 ¼ 12:35 kN=m 988 15 Vertical Loads on Multistory Buildings The earthquake loads are listed below. Fig. E15.2d Computer generated loadings. We analyze a 2D model of the rigid frame for gravity loading and earthquake loadings separately. This approach is possible because the analysis is linear. The values of the forces (axial, shear, moment) occur in the first story. Results for the bearing, shear, and axial forces corresponding to the two choices for column properties are listed and plotted below. ... Earthquake Case(1) ... Gravity Case(2) ... Earthquake Case(2) ... 15 Vertical Loads on Multistory Buildings Fig. E15.2e Shear, moment, axial force diagrams—gravity loading 15.4 A Case Study: Four-Story Building 989 Fig. E15.2f Shear, moment, axial force diagrams—earthquake loading Fig. E15.2g Lateral displacement of the floors. We note that there is only a small difference in the force magnitudes when the column inertia values are doubled. The main effect is on the lateral displacement, which is reduced by a factor of 2. 15.4 A Case Study: Four-Story Building In this section, we illustrate the computation of the design parameters for two typical structural systems, a rigid frame and a partially braced frame. We also use the case-based procedures to estimate the required design parameters which provide an estimate of the relative efficiency of the two systems. 15.4.1 Building Details and Objectives The building is a four-story steel frame building with a green roof. Figure 15.14 shows the typical floor plan and elevation views. The rigid flooring system transmits the gravity load through the floor beams oriented in the N–S direction (one-way action). The loading and member data are as follows: • Floor load = 0.055 kip/ft2 • Floor live load = 0.07 kip/ft2 • Roof dead load = 0.18 kip/ft2 990 15 Vertical Loads on Multistory Buildings • Roof live load = 0.02 kip/ft2 • Global wind loads acting in the N–S and E–W directions are shown in Fig. 15.15. They correspond to a peak wind speed of 80 mph for a building located in Boston, Massachusetts. • Weight of exterior walls is 1.1 kip/ft • Based on economical considerations, All floor beams are the same size. – All the girders in the N–S direction are the same size. – All floor/roof beams in the E–W direction are the same size. – All the braces have the same size. • The following combinations of loads for strength design are to be considered: 8 1:4wD ≥ wu ¼ 1:2wD þ 0:5wLroof ... • The following limits are required by the serviceability constraint: Lbeam > Lbeam for ΔL ≤ ... Limit beam deflection to L/360 H building Limit building drift to 300 Case (1): The structure is a braced frame, i.e., all the connections between beams and columns are pinned. Case (2): The structure is a rigid frame in the N–S direction and braced frame in the E–W direction. 15.4 A Case Study: Four-Story Building Fig. 15.14 Typical floor plan and elevation views—case study. (a) Plan. (b) N–S elevation—Section A-A. (c) E–W elevation—Section B-B 15.4 A Case Study: Four-Story Building 991 Fig. 15.15 Global wind loads. (a) Plan. (b) N–S. (c) E–W • The following limits ... 15.4.2 Case (1) Frames Are Braced in Both N–S and E–W Directions All the connections are pinned. The lateral load is carried entirely by the bracing. We use the K bracing schemes shown in Fig. 15.16. 992 15 Vertical Loads on Multistory Buildings Fig. 15.16 Braced frame configuration. (a) Plan— braced in both directions. (b) N–S elevation—braced frames A-A, D-D, G-G. (c) E–W elevation braced frames 1-1, 4-4 The braces have equal stiffnesses and the floors are rigid. Therefore the global wind load is distributed equally between braces (see Chap. 14). The column load is purely axial since the members are all pinned. We establish the column load per floor working with the tributary areas shown on the sketch below. We assume the wind load is carried by the bracing. Bracing Design The bracing is governed by the lateral loading. We establish the lateral forces in the diagonals. 15.4 A Case Study: Four-Story Building 993 The combined wind load on the N–S side is H = 300. We determine the inter-story displacement in terms of the brace deformation. Noting the equations presented in Sect. 11.4.3, we solve for the required area. ... The diagonal members may be subjected to either tension or compression depending on the direction of the wind. The maximum axial force in the diagonal members is 19.7 kip. Applying the appropriate load factor, the design value is Pu ¼ 19.7(1.6) ¼ 31.5 kip. ... The column design is based on the maximum axial load. ... Using a design axial load of Pu = 31.5 kip, and an effective length of 19.2 ft, and the required area based on the design axial load of 0.48 in.2, selects a cross-sectional area and we choose a section. Also, the beam deflection limit ... 15.4 A Case Study: Four-Story Building 995 We show the required design for the N–S direction. ... Interior Columns The column load is based on the tributary areas. The loads are PD ¼ 20ð30Þð0:18 þ ... Þ ¼ 207 kip PL roof ¼ ... ¼ 12 kip PL ¼ ... ¼ 126 kip PNSWind ¼ 29:4 kip Evaluating the following load combinations 8 1:4PD ¼ 290 kip > > Pu ¼ 1:2PD þ 1:6PL þ 0:5PLr ¼ 456 kip ... Selecting a section for the columns with Pu = 456 kip and effective length of 12 ft. 15.4.2.1 Interior Beams Interior Floor Beams (30 ft span): wDfloor ¼ 0:055ð20Þ þ ... 15.4.2.2 Summary for Case (1) The relevant design parameters for the braced frame are listed below. ... 15.4.3 Case (2) Frames Are Rigid in the N–S Direction But Remain Braced in the E-W Direction Figure 15.17a shows a plan view of this scheme. The frames in the N–S direction are now rigid frames. Our objective here is to compare the response of an individual rigid frame to the response of a braced frame. We use the same procedure to estimate the design values for the beams and columns for the rigid frame. Fig. 15.17 Rigid frame scheme. (a) Plan. (b) Rigid frame elevation—N–S, wind loading. (c) E–W elevation—braced frames 1-1, 4-4; wind loading 998 15 Vertical Loads on Multistory Buildings 15.4.3.1 Strategy for N–S Beams and Columns We specify moment connections between the beams and the columns in the N–S direction. We assume that the lateral wind load will be carried by the seven rigid frames, because the floor slabs are rigid and the frames have equal stiffnesses. The E-W direction remains the same as the beams in this direction are pin ended. Since the beams in the N–S direction are now rigidly connected to the columns, end moments will be developed in the beams. The net effect is a reduction in the maximum moment in the beams. For a first estimate, assuming full fixity, the peak moment reduces from wL2/8 to wL2/12, a reduction of 33 %. It follows that the beams will be lighter; however, the columns will be heavier since they now must be designed for both axial force and moment. Wind loading introduces end moments in the beams and columns. We use the portal method (see Chap. 11) to estimate these values. The results are shown on the sketch below. 15.4 A Case Study: Four-Story Building 999 15.4.3.2 Estimated Properties: Beams We estimate the design moment for roof and floor beams based on the following combination of factored moments: Roof: 8 wD L2 > ¼ 378 kip ft 1:4 > > 12 > > = L2 M ¼ 372 kip ft ð þ Þ 1:2w u ð 1:6w L Þ > 12 > > > ; 0:5w þ L 1:6M D L wind ¼ 343 kip ft 12 Floor: 8 wD L2 > > ¼ 115:5 kip ft 1:4 > > 12 > = L2 M ð 1:2wD þ 1:6wL Þ ¼ 267 kip ft u > 12 > > > ; 0:5w þ L þ 1:6M D L wind ¼ 209 kip ft 12 As a first estimate, we use tributary areas to estimate the axial load in the columns due to dead and live loads. The most critical load combinations for the columns are (Pu ¼ 1:2PD þ 1:6Pwind þ 0:5PLfloors þ 0:5PLroof 317 kip (Mu 1:6Mwind 1:6ð52:2Þ 84 kip ft Pu ¼ 1:2PD þ 1:6PLfloors þ 0:5PLroof 461 kip Mu 10 kip ft Based on the above estimated force values, we select the following cross-sectional properties. 8 I col ¼ 272 in:4 Acol ¼ 14:4 in:2 > > < I beam=floor ¼ 1070 in:4 Abeam=floor ¼ 19:1 in:2 > > : I beam=roof ¼ 1830 in:4 Abeam=roof ¼ 24:3 in:2 Determining the actual properties is an iterative process. We expect the beam sizes to decrease, and the column sizes to increase as the iteration proceeds due to the shift from braced frame to rigid frame. We orient the cross sections such that the bending occurs about the strong axis as indicated on the sketch below. 1000 15 Vertical Loads on Multistory Buildings 15.4.3.3 Live Load Patterns We determine the live load patterns for the maximum positive and negative moments for the beams and for the maximum axial force for columns and then analyze these structures under the combined dead, live, and wind loads. The wind loads are defined in Fig. 15.17b. Figure 15.18 shows the live load patterns for maximum moments in beams, and axial forces in columns. Live Load Patterns for Positive Moment—Beams: There are two load patterns for maximum positive moment at mid-span of the beams. Negative Moment Live Load Patterns—Beams: There are eight loading patterns for maximum negative end moments of the beams. Fig. 15.18 (a) Positive live load (LL) moment patterns (1)–(2). (b) Negative live load (LL) moment patterns (3)–(10). (c) Live load patterns for axial force in column (11)–(12) 15.4 A Case Study: Four-Story Building Fig. 15.18 (continued) 1001 1002 15 Vertical Loads on Multistory Buildings Fig. 15.18 (continued) Axial Force Live Load Patterns—Columns: The following two patterns establish the peak values of the column axial forces. 15.4.3.4 Discrete Moment Envelope Plot-Live Load Using a computer software system, results of the analyses for the ten live load patterns defined in Fig. 15.18a, b are used to construct the discrete moment envelope plots shown in Fig. 15.19. These plots show the peak positive and negative moments at various sections along the spans generated by the different loading patterns. The absolute peak values are summarized in Fig. 15.20. Fig. 15.19 Peak positive and negative discrete moment envelopes—live load 15.4 A Case Study: Four-Story Building 1003 Fig. 15.20 Absolute maximum positive and negative moments in the beams due to pattern live loading (kip ft) The factored discrete envelopes are plotted in determine revised values for the cross-sectional 8 I col ¼ 421 in:4 > > < I beam=floor ¼ 890 in:4 > > : I beam=roof ¼ 1140 in:4 Fig. 15.21. Using this updated information, we Acol ¼ 17:7 in:2 Abeam=floor ¼ 16:2 in:2 Abeam=roof ¼ 16:2 in:2 Lastly, using these properties, we generate updated values for the design moments shown in Fig. 15.22. Since the columns are subjected to both axial action and bending, we need to scan the results for the individual loadings and identify the loadings that produce the maximum axial force and the maximum moment in the columns. Carrying out this operation, we identify the combinations for columns C1 and C5 listed in Fig. 15.23. Given these design values, one generates new estimates for the cross-sectional properties. If these new estimates differ significantly from the previous values, the analyses need to be repeated since the results are based on the relative stiffness of the beams and columns. It turns out for this study that the initial estimates are sufficiently accurate. 15.4.3.5 Summary Yc (see 2) N–S roof beam N–S floor beam E–W beams Columns E–W braces Mu ¼ 442 kip ft Mu ¼ 336 kip ft same as case(1) Pu ¼ 490 kip Pu ¼ 306 kip or Mu ¼ 10 kip ft Mu ¼ 111 kip ft same as Cg (1) 1004 15 Vertical Loads on Multistory Buildings Fig. 15.21 (a) Discrete moment envelope–factored load combination. (b) Discrete axial force envelope–factored load combination. (c) Discrete shear envelope–factored load combination 15.4 A Case Study: Four-Story Building 1005 Fig. 15.22 Maximum and minimum design moments (kip ft) Fig. 15.23 Critical axial load–moment combinations for Columns C1 and C5. (a) Exterior column. (b) Interior column 15.4.4 Discussion The following table contains the design values corresponding to the two cases. N–S roof beam N–S floor beam E–W beams Columns Case (1) Braced Mu ¼ 567 kip ft Mu ¼ 336 kip ft Mu ¼ 77 kip ft Pu ¼ 456 kip E–W beams N–S beams Pu ¼ 24 kip Pu ¼ 31:5 kip Case(2) Rigid N–S beam Mu ¼ 442 kip ft Mu ¼ 336 kip ft Mu ¼ 77 kip ft Pu ¼ 490 kip Pu ¼ 306 kip or Mu ¼ 10 kip ft Mu ¼ 111 kip ft Pu ¼ 24 kip Not required Comparing values, we see that the N–S rigid frame structure is more efficient in the sense that its design values are less, and therefore the required cross-sections are lighter. However, the lateral displacements will be greater. 1006 15.5 15 Vertical Loads on Multistory Buildings Summary 15.5.1 Objectives • To describe how gravity loads are applied to the floors of multistory buildings. • To show how Mu¨ller-Breslau principle can be applied to establish critical patterns of live gravity loading for the peak bending moments in rigid frames. • To present a case study which integrates all the different procedures for dealing with dead, live, wind, and earthquake loads. 15.5.2 Key Concepts • The floor slabs in concrete buildings are cast simultaneously with the supporting beams. The type of construction provides two possible load paths for gravity loads. Which path dominates depends on the relative magnitude of the side dimensions. The terms one-way and two-way actions are limiting cases where (1) one side is large with respect to the other side and (2) the sides are of the same order of magnitude. • Gravity loading produces only positive moment in the beams of a braced frame. • Positive moment at mid-span in the beams of a rigid frame is due only to gravity loading. • Negative moment in the beams is generated by both gravity and lateral loads. 15.6 Problems Problem 15.1 4 3m 4 3m 4 3m 6m 6m Using Mu¨ller-Breslau Principle, determine the live loading patterns (uniformly distributed member load) that produce the peak values of positive moment at point A (mid-span) for each story. 15.6 Problems Problem 15.2 10 ft B 10 ft B 10 ft B 25 ft 25 ft 25 ft Using Mu¨ller-Breslau Principle, estimate the loading patterns (uniformly distributed member load) that produce the peak value of negative moment at B for each story. Check the results using a software package. Take Ic ¼ 150 in.4 for all the columns and Ig ¼ 300 in.4 for all the beams. Problem 15.3 Using Mu¨ller-Breslau Principle, determine the loading patterns (uniformly distributed member load) that produce the peak value of negative moment at B for each story. Check the results using a software package. Take Ib ¼ 300(10)6 mm4 for all the beams and Ic ¼ 100(10)6 mm4 for all the columns. B 3m B 3m B 4m 8m 6m 9m Problem 15.4 For the frame shown below (a) Using Mu¨ller-Breslau Principle, sketch the influence line for the positive moment at A and the negative moment at B. 1008 15 Vertical Loads on Multistory Buildings (b) Use a software package to determine the maximum values of these moments due to a uniformly distributed live load of 30 kN/m and a uniformly distributed dead load of 20 kN/m. Take Ic ¼ 100(10)6 mm4 for all the columns and Ib ¼ 200(10)6 mm4 for all the beams. Problem 15.5 For the frame shown below (a) Using Mu¨ller-Breslau Principle, sketch the influence lines for the positive moment at A and the negative moment at B. (b) Use a software package to determine the maximum values of these quantities due to a uniformly distributed live load of 1.8 kip/ft and a uniformly distributed dead load of 1.2 kip/ft. Take Ic ¼ 480 in.4 for all the columns and Ib ¼ 1000 in.4 for all the beams. 10 ft A 12 ft 30 ft 20 ft 20 ft 30 ft 15.6 Problems 1009 Problem 15.6 12 ft 12 ft 12 ft 15 ft 30 ft 20 ft 30 ft Consider the typical frame defined above. Assume the bay spacing is 20 ft. (a) Determine the floor loads per bay due to an earthquake of intensity Sa ¼ 0.3g. Assume the following dead weights. Roof load ¼ 0.08 kip/ft2 and floor load ¼ 0.06 kip/ft2. (b) Estimate the column shear forces due to this earthquake. (c) Estimate the column shear forces due to both gravity and earthquake. Problem 15.7 Consider the frame shown below. Assume a uniform gravity live load for the beams. (a) Describe how you would apply Mu¨ller-Breslau principle to establish the loading pattern for the compressive axial load in column A. (b) Compare the axial load in column A of the pattern loading to the uniform loading on all members. Consider all the girders to be of the same size and all the columns to be of the same size. Assume Ibeam ¼ 2.5Icolumn and ¼ 1.2 kip/ft. Use computer software. 1010 15 Vertical Loads on Multistory Buildings Problem 15.8 Discuss the function of the structure, abcdef. How would you determine the axial force in member ad due to gravity loading on the floors of the multistory rigid frame building shown below. Investigate how the internal forces vary in the angle α ranging from 0 to 20 , considering H constant. Is there a limiting value for α? 15.6 Problems 1011 Problem 15.10 Consider the structures shown below. All the members are pinned at their ends. (a) Determine expression for the axial forces in the diagonal members. (b) Determine the horizontal and vertical displacements of the nodes 1 and 2. (c) Extend the analysis to the structure shown below. This structure is called a DIAGRID structure. Problem 15.11 For the structure shown below, assume the floors are flexible and the flooring system transits the gravity loading to the floor beams in the N–S direction (one-way action). Assume all the columns are the same size. Ibeam ¼ 3Icolumn, Floorgravity ¼ 175 kN/m2. Compare the maximum forces in beams and columns caused by a uniform gravity load and with the following cases. 1. The structure is considered to be a braced frame i.e., all the connections between beams and columns are pinned both in N–S and E–W direction. 2. The structure is considered to be a rigid frame in the N–S direction and a braced frame in the E–W direction, i.e., all the connections between beams and columns are moment connections in N–S direction, but connections in the E–W direction remain pinned. 1012 15 Vertical Loads on Multistory Buildings Inelastic Response of Structures 16 Abstract The conventional design approach works with factored loads and reduced material capacities such as strength. When subjected to service loads, the structure is detailed such that it behaves elastically. For extreme loadings, the structure is allowed to experience a limited amount of deformation beyond the elastic limit. This deformation is called "inelastic" since in contrast to elastic deformation, when the loading is removed, the structure does not return to its original position. Up to this point in the text, we have assumed the behavior to be elastic. In Chap. 10, we included geometric nonlinear effects but still assumed elastic behavior. Here, we introduce an additional effect, inelastic behavior. We start with an in-depth discussion of the stress-strain behavior of structural steels and concrete, apply these ideas to beams subjected to inelastic bending, and then develop an analysis procedure to determine the inelastic response of frame-type structures. This approach allows one to estimate the "maximum" loading that a structure can support, i.e., the "limit load." Examples illustrating the influence of inelastic behavior on the ultimate capacity of structures are included. 16.1 Stress-Strain Behavior of Structural Steels Steel and concrete are the two most popular construction materials. Steels with low carbon content are usually referred to as "structural" steels since they are used primarily to fabricate structural elements such as W, T, and I shapes. Structural steels have desirable properties such as strength, uniformity, weldability, and ductility. The latter property is related to the ability of structural steels to experience significant deformation prior to fracture. Figure 16.1 shows a typical stress–strain plot for a mild (low carbon) steel. There are four distinct deformation zones: elastic; yielding; strain hardening; and necking. The stress remains constant as the 0 specimen yields until the strain, εy , is reached. Beyond this level, strain hardening occurs and the stress increases to its "ultimate" value and then decreases to the value at which rupture occurs. The yield zone for mild steel is relatively large, on the order of 20 times the yield strain εy. Another # Springer International Publishing Switzerland 2016 J.J. Connor, S. Faraji, Fundamentals of Structural Engineering, DOI 10.1007/978-3-319-24331-3_16 1013 1014 16 Inelastic Response of Structures Fig. 16.1 Typical stress–strain curve for mild structural steel Fig. 16.2 Steel properties—stress–strain curves measure is the rupture strain εr. By definition, εr is the fractional change in length of the specimen and is usually expressed as a percentage. For mild steel, the percent elongation is about 24 %. Ductility εr refers to the ability to deform plastically without fracturing and is measured by the ratio . This ratio is a key component of toughness. 1 The last measure of interest is the toughness, defined as the energy required to fracture the material. Toughness is equal to the area under the stress–strain curve to fracture, and is usually expressed as a multiple of R, the area under the linear portion of the stress–strain curve. Figure 16.2 lists the stress–strain plots for a range of common structural steels. Note that as the yield stress and the ultimate stress increase, the yield zone and the rupture strains decrease. There is a 16.2 Inelastic Moment–Curvature Relationships 1015 trade-off between ductility and strength. The yield zones for the two lower strength steels are 0 εy essentially equal γ 13 whereas the higher strength steels shift directly from elastic behavior into strain hardening. 16.2 Inelastic Moment–Curvature Relationships The fact that structural steels can experience significant deformation before rupturing is the basis for developing an analysis procedure for tracking the response as the structure passes from the elastic range through the yielding zone up to rupture. The starting point for the analysis is establishing the moment capacity of a beam subjected to inelastic bending. Assuming a cross section remains a plane, the extensional strain varies linearly with distance from the centroidal axis. ε ¼ χ y ð16:1Þ where χ is the curvature of the centroidal axis. Given χ, one computes ε and then determines the stress using a nonlinear stress–strain relation such as shown in Fig. 16.1. ε ! σ ¼ f ðεÞ ¼ f ðχ yÞ ð16:2Þ The moment is defined by the following integral, ð M ¼ ðyσÞdA ¼ ð Mðχ Þ ð16:3Þ This integral is usually evaluated numerically; one ranges over χ and constructs a plot of M vs. χ. The exact form depends upon the assumed stress–strain relationship and the shape of the cross section. In order to obtain an analytical solution, the stress–strain curve is usually approximated with a linear model. The simplest model for steel is based on the assumption of elastic perfectly plastic fracture behavior; the structure behaves as an elastic perfectly plastic material which is illustrated. Figure 16.3 illustrates this behavior. Fig. 16.3 Elastic perfectly plastic fracture (EPPF) model 1016 16 Inelastic Response of Structures Using this model, and applying (16.1)–(16.3) to a rectangular cross section, one obtains the plot shown in Fig. 16.4. 2 bh My ¼ σy 6 ( ) 3 1 εy 2 Mr ¼ My 1 2 3 εr χy ¼ 2εy h χy ¼ 2εr h Fig. 16.4 Moment–curvature plot—rectangular sector and EPPF material The outer fiber fractures at χ ¼ χr , resulting in a discontinuity in the derivative. For χ > χ r , yielding progresses over the section and the moment capacity rapidly decreases. Wide flange sections behave differently since the moment is carried primarily by the flanges. Fracture in the flanges results in a significant drop in moment capacity, especially when the sections are thin-walled since, in this case, the fracture occurs simultaneously through the flange thickness. A typical plot is shown in Fig. 16.5. Fig. 16.5 Moment–curvature plot—thin walled wide flange section and EPPF material From a computer implementation perspective, it is convenient to represent the moment–curvature relations with the bilinear approximation shown in Fig. 16.6. Nonlinear analysis is required to evaluate the load-deflection behavior. With this type of approximation, when yielding occurs, one replaces the elastic stiffness with a "linearized" tangent stiffness. The simplest possible strategy is to 16.2 Inelastic Moment–Curvature Relationships 1017 Fig. 16.6 Bilinear moment–curvature models. (a) General bilinear model and (b) EPPF model Fig. 16.7 Compressive stress–strain curves for concrete work with an elastic perfectly plastic fracture model, i.e., to assume the tangent stiffness is negligible for χ y < χ < χ r . This behavior is introduced by inserting a hinge at the location of the cross section and applying a constant moment equal to My. When χ r < χ, the constant moment is removed, resulting in no moment capacity at the section. Although the discussion has been focused on steel, the concept of a bilinear moment–curvature model is also appropriate for concrete. The behavior of concrete differs from that of steel in that the stress–strain relationship for concrete exhibits strain softening (a reduction in stress beyond the peak stress) and the peak strain is several orders of magnitude lower than that of steel. Typical plots are shown in Fig. 16.7. Values corresponding to crushing of the concrete and to development of a strain of 0.0004. Assuming the strains vary linearly over the cross section; one can construct the moment–curvature relationship. A typical plot for an under-reinforced section is shown in Fig. 16.8. For computer-based analysis, one uses the nonlinear form. For hand computation, this form is approximated with 0 an elastic perfectly plastic fracture model where My is considered to be the ultimate moment capacity. A detailed discussion of this topic is contained in Winter and Nilson [1]. 1018 16 Inelastic Response of Structures Fig. 16.8 Moment–curvature relationship—concrete beams 16.3 Limit Analysis: A Simplified Approach Given the properties of a structure and a particular loading distribution, it is of interest to establish the peak magnitude of the loading that the structure can support, i.e., the limit of the loading. In what follows, we describe a procedure based on using an elastic perfectly plastic fracture model. Starting at a low load level, one carries out an elastic analysis and identifies the section where the bending moment is a relative maximum. The loading is then scaled up such that the magnitude of the moment at that section equals the moment capacity. In that section, a hinge is inserted and a set of self-equilibrating concentrated moments equal to My are applied at the section. The modified structure is then examined with respect to its stability, i.e., its capacity to support additional load. If stable, the loading is increased until another cross section reaches its moment capacity. The structure is again checked for stability and, if stable, a hinge and the assumed set of moments are inserted. The process is continued until the modified structure is unstable. The following examples illustrate this process. Example 16.1 Given: A simply supported beam subjected to a uniformly distributed loading defined in Fig. E16.1a. Fig. E16.1a Determine: The load capacity. Solution: The peak moment occurs at mid-span. 16.3 Limit Analysis: A Simplified Approach 1019 Mmax ¼ wL2 8 We increase the load until Mmax ¼ My , wL L2 8 ¼ 8My wl ¼ L2 My ¼ The Modified structure is shown in Fig. E16.1b. Fig. E16.1b This load level is the ultimate load. Any increase in load will cause the structure to collapse downward since it has no capacity to carry any additional load. Therefore, w1 ¼ wmax . Example 16.2 Given: The fixed-ended beam subjected to a uniform loading defined in Fig. E16.2a. Fig. E16.2a Determine: The load capacity. Solution: The peak moment occurs at A and C. Therefore, yielding will first occur at these sections. We increase the loading until Mmax ¼ My . 1020 16 Inelastic Response of Structures w1 L2 12 ¼ 12My w1 ¼ L2 The modified structure is shown in Fig. E16.2b. Fig. E16.2b The next critical section is at mid-span. Applying an incremental load, Δw, increases the moment at mid-span. Fig. E16.2c Then, superimposing the results for the two cases leads to the peak moment at mid-span which is set equal to My. ΔwL2 w1 L2 þ ¼ My 8 24 Substituting for w1 , ΔwL L2 8 ¼ ¼ 1 ¼ My 12 3 My 2 8 leads to Δw ¼ 4 My L2 16.3 Limit Analysis: A Simplified Approach 1021 Lastly, wmax ¼ w1 þ Δw ¼ 16 My L2 The final "limit state" is shown in Fig. E16.2d. The structure is unstable for any additional transverse loading. Fig. E16.2d Example 16.3 Given: The two-span beam shown in Fig. E16.3a. Assume EI is constant. Fig. E16.3a Determine: The limit state. Solution: The moments at sections B and C are relative maxima; of these values, the value at B is critical, so yielding will occur first at section B. 13 PL 64 6 MC ¼ PL 64 MB ¼ Load level 1: We set MB equal to My and insert a hinge at B. 13 P1 L ¼ My 64 ¼ 64 My 13 L 1022 16 Inelastic Response of Structures At this load level, the modified structure is as shown in Fig. E16.3b. Load level 2: We apply an incremental load, ΔP, to the modified structure and increase the loading until the moment at C equals My. Fig. E16.3b Fig. E16.3c 6 ΔPL L 6 þ P1 L þ ΔP ¼ My 64 64 4 Substituting for P1 obtains ΔP ¼ 4 My 13 L Finally Pmax ¼ P1 þ ΔP ¼ 78 6 My My 13L L 16.3 Limit Analysis: A Simplified Approach 1023 The limit state has hinges at B and C. Fig. E16.3d ——————————————————————————— The procedure followed in the above examples involved applying the loading incrementally and analyzing the structure at each load level. When yielding is reached at a particular load level, the structural stiffness is modified by inserting a hinge (i.e., zero rotational stiffness) at the yielded section. The loading process is continued until the structure becomes unstable. From a structural engineering perspective, instability occurs when the structural stiffness associated with the complete structure vanishes. When we previously derived the response, we must also check that the curvature at a yielded section does not exceed the "rupture" or "crushing" value. When this occurs, the moment capacity is set to zero. We did not carry out this computation in the above examples since it requires a computer-based procedure in order to obtain reliable results. Later in the chapter, we describe a procedure based on a finite element beam discretization combined with an incremental nonlinear analysis strategy. Most modern structural software systems have this type of capability. An alternative hand calculation approach to establishing the limit state is based on assuming a pattern of hinges that corresponds to a limit state and evaluating the corresponding load magnitude. One starts with the elastic moment diagram and identifies the sections where the moment is a relative maximum. At the limit state, the number of plastic hinges is equal to 1 plus the number of degrees of freedom. The load magnitude can be obtained either by applying the equilibrium equations or using the principal of virtual work which is an equivalent statement of equilibrium. The latter approach is generally more convenient. We illustrate this procedure with the following examples. Example 16.4 Given: A two-span beam shown in Fig. E16.4a. Fig. E16.4a 1024 16 Inelastic Response of Structures Determine: The limit state. Solution: The critical state has two hinges. The moment diagram, we locate them at Points B and C. Fig. E16.4b Fig. E16.4c We use the principle of virtual work to establish the expression for P. Introducing a virtual displacement at B and evaluating the work done by P and the plastic moments leads to PΔu My Δθ1 þ My Δθ2 The virtual rotations and the vertical displacement terms are related by Δ u ¼ L 2 Δθ1 Δθ2 ¼ 2 Δ u L Substituting for the Δθ terms, the work equation reduces to My 1 Δu P 2My ¼ L Δu 2 L Pu 8 My P ¼ La This must be satisfied for arbitrary Δu. Then P ¼ My 1 2 þ a La 16.3 Limit Analysis: A Simplified Approach 1025 Example 16.5 Given: A two-story frame shown in Fig. E16.5a. Fig. E16.5a Determine: The limit state. Solution: For this frame, all the relative maxima occur at the nodes. Fig. E16.5b Plastic hinges will develop at the end of the members. When designing the members, most design codes require that one selects sectional properties such that yielding occurs only in the beams. We assume that condition is satisfied here, and work with the limit state shown below. Fig. E16.5b Introducing a virtual displacement, Δ, the work terms are P Δu 4My Δθ ¼ 4 0 2 2 þ 5 Δu PΔu 4My Δθ ¼ 4My Δθ ¼ 4 20h þ 8M y Pmax ¼ 5h PΔu þ In the above example, we assumed a particular plastic hinge pattern. Other patterns are also possible, each with a different limit load. One needs to examine all possible patterns in order to identify the minimum critical load. This operation is not feasible using hand computation for a complex structure. One needs to employ a computer-based nonlinear analysis scheme which generates the load displacement response allowing for the formation of plastic hinges up to the load level at which collapse is imminent. A particular nonlinear analysis scheme is described in the next section. 1026 16 Inelastic Response of Structures 16.4 Nonlinear Analysis Scheme We illustrate this method using the structure analyzed in Example 16.4. The first step involves discretizing the structure with a combination of elastic and plastic finite elements. A refined mesh is used in those zones where the moment is a relative maximum, such as adjacent to interior nodes and concentrated loads. Since the extent of plastic yielding is not known initially, one needs to iterate, starting with a single plastic element and adding additional plastic elements if necessary. This process is illustrated in Fig. 16.9. The material behavior within a plastic segment is assumed to follow the bilinear moment–curvature model defined in Fig. 16.10. When χ < χ y , the behavior is elastic, and the elastic stiffness KE applies. When χ > χ y , the behavior is inelastic and the reduced stiffness, kt, is used. When χ > χ r , the moment is set equal to zero, i.e., the section is considered to have ruptured. The derivation of the elemental equations is based on the Principle of Virtual Displacements [2] (see Fig. 16.11. Nodes are located at each end, and the nodal displacement measures are the translation and rotation. Introducing matrix notation, these displacement measures are expressed as v1 v2 U1 ¼ U2 ¼ θ1 θ2 The transverse displacement is approximated as v ðx Þ ¼ Φ1 U 1 þ Φ2 U 2 where Φ1 and Φ2 contain interpolation functions. Fig. 16.9 Plastic element discretization. (a) Initial mesh and (b) expanded mesh Fig. 16.10 Bilinear model ð16:4Þ 16.4 Nonlinear Analysis Scheme 1027 Fig. 16.11 End displacements 3 x2 x Φ1 1 3x 2 þ 2 x 3x þ 2 2 þ L L L L 3 x2 x Φ2 ¼ 3x 2 2 x3 2x 2 þ L L ð16:5Þ Differentiating twice leads to the curvature χ ¼ v, xx ¼ Φ1 , xx U 1 þ Φ 2 , xx U 2 6 12x 4 6x Φ1 , xx ¼ 2 þ 3 2 þ L L 6 12x 2 6x Φ2 , xx ¼ 2 þ 3 2 þ L L L L ð16:6Þ The end forces are determined with the following virtual work expression: ð δ M δχ dx PT 1 δU1 þ PT 2 δU 2 ¼ for arbitrary δU 1 and δU2 . Noting δχ ¼ v, xx ¼ Φ1, xx δU 1 þ Φ2 , xx δU 2 ð16:7Þ and expanding (16.7) result in P 1 ¼ ΦT 1, xx M dx δ P 2 ¼ ΦT 2, xx M dx ð16:8Þ ð16:9Þ When the behavior is inelastic, a numerical integration scheme, such as a 2 point Gaussian approximation, is used to evaluate the integrals. The end forces are determined using Fig. 16.10. The system equations for this problem is written as PE ¼ Pi ð16:10Þ where PE contains the external loads applied to the nodes and Pi contains the nodal loads due to the member end actions which are functions of the nodal displacements. The solution scheme proceeds as follows. Suppose U is known and we want to establish the solution for the response due to PE . The static error is E ¼ ¼ P E PI ð16:11Þ We correct the error by introducing an incremental displacement ΔU i.e., by expanding the unbalanced nodal forces. ΔPI ¼ ΔPE ð16:12Þ where Kt represents a "tangent" stiffness matrix for the structure The incremental equilibrium equation takes the form K t i ΔU i ¼ PE i PI i ð16:14Þ One cycles on (16.14) until successive value of ΔU agree to a specified tolerance. Instability occurs when Kt is singular. One determines ΔPI by operating on (16.9). For example, noting δ ΔP1 ¼ Φ T, xx ΔM dx ΔM ¼ Kt Δχ for χ y < χ < χ r ΔM ¼ 0 for χ > χ r ΔM ¼ kt Δχ for χ < χ y ð16:15Þ Δχ ¼ Φ1, xx ΔU 1 þ Φ2, xx ΔU 2 leads to δ ΔP1 ¼ ΦT, xx k* ΦT, xx ΔU 1 þ Φ2, xx dx ΔU 2 where k* ¼ kE , kt , or 0 depending on the value of χ. One applies the external load in increments and generates the incremental response using (16.14). Nonlinear structural analysis computer software systems use this approach to generate the complete nonlinear load-displacement response for a typical structure, i.e., it determines the order and location of plastic hinges as the load is increased, and the limit load. Most commercial structural software have this capability. The following examples illustrate nonlinear analysis. Example 16.6 Given: The portal frame shown in Fig. E16.6a. Assume the lateral load P is due to seismic excitation. The material is steel, σ y ¼ 50 ksi, and the frame has a weight of 4.17 kip/ft. Fig. E16.6a 16.4 Nonlinear Analysis Scheme 1029 Determine: The inelastic response of the frame to seismic excitation. Determine the value of P and Sa. Solution: The analysis of gravity loaded frames subjected to lateral loading is referred to as a pushover analysis. One common application is to establish the capacity of a frame for seismic excitation. One applies the lateral load incrementally and generates the nonlinear response up to the onset of instability. The pushover analysis was done using computer software [3] and the result is plotted in Fig. E16.6b. Fig. E16.6b Pushover results, P vs. joint displacement Using the materials presented in Sect. 14.2.2, one can relate P to the spectral acceleration. For a single degree of freedom system, this relationship reduces to P mSa where m is the lumped mass and Sa is the spectral acceleration. Pmax ¼ 32 kip 4:17040 166:8 ¼ g g Pmax Sa ¼ 0:19 g m 1030 16 Inelastic Response of Structures Example 16.7 Given: The three-story frame defined in Fig. E16.7a. Consider that the gravity floor loading to be constant. The lateral load is due to seismic excitation. The material is steel, σ y ¼ 50 ksi, and the gravity floor load ¼0.75 kip/ft. Fig. E16.7a Determine the lateral displacement of point A versus P and the limiting value of Sa. Solution: We use Equation (14.9) specialized for this frame. P ¼ mΓSa where m is the mass of a typical floor, m ¼ 0:75ð60Þ 45 ¼ g g 9 the influence factor Γ ¼ 16.5 Summary 1031 Fig. E16.7b Pushover results, P vs. joint displacement Then Sa ¼ Pmax 115 ¼ ¼ 1:98g 45 9 mΓ g 16.5 7 Summary 16.5.1 Objectives • Describe the different regions of the stress–strain behavior of structural steels and concrete: elastic; inelastic. • Extend the moment–curvature relationship to the inelastic range and estimate the moment capacity of a beam subjected to inelastic bending. • Present analysis procedures for determining the maximum external load that a structure can support using: (a) hand calculation methods and (b) finite element computer-based methods. This general topic is called "Limit Analysis." • Illustrate how the Limit Analysis is applied to simple rigid frames. 16.6 Problems Problem 16.1 Determine the load capacity. Problem 16.2 Determine the load capacity. Inelastic Response of Structures 16 Problems Problem 16.3 Determine Pmax as a function of α. Problem 16.4 Determine an expression for Pmax. Problem 16.5 Generate the plot of P vs. u for the frame shown. Consider w as a dead loading. Assume α ¼ 1.2. 1034 16 Inelastic Response of Structures Problem 16.6 Using computer software, generate the plot of uA vs. P, and estimate Pmax. Take w ¼ 1.5 kip/ft, h ¼ 12 ft, and L ¼ 30 ft. The material is steel, σ y ¼ 50 ksi. The exterior columns are aW16 100, and all the beams are W24 131. References 1. Nilson AH. Design of concrete structures. 14th ed. New York: McGraw Hill; 2013. 2. Bathe KJ. Finite element procedures. 1st ed. Prentice Hall: Upper Saddle River; 1995. 3. GTSTRUDL. Intergraph Corporation PP&M, Norcross, GA, USA.